Strange attractors of forced one-dimensional systems: existence and geometry

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Kurzdarstellung

Nichtglatte Sattel-Knoten Bifurkationen von einparametrischen Familien quasiperiodisch getriebener Dynamischer Systeme auf den reellen Zahlen erzeugen seltsame nichtchaotische Attraktoren.

In dieser Arbeit konstruieren wir eine Klasse von Familien, die nichtleeres Inneres in der C^2-Topologie hat und deren Elemente nichtglatte Sattel-Knoten Verzweigungen durchlaufen. Innerhalb dieser Klasse untersuchen wir die Geometrie der zugehörigen seltsamen Attraktoren, indem wir verschiedene fraktale Dimensionen bestimmen. Wir erhalten unter anderem, dass die Hausdorff- und die Boxcounting-Dimension unterschiedliche Werte annehmen. Darüber hinaus zeigt sich, dass die minimale Menge am Verzweigungspunkt maximal invariant ist.

Unsere Ergebnisse decken sowohl den zeitdiskreten wie auch den zeitkontinuierlichen Fall ab. Explizite Beispiele unterstreichen die Anwendbarkeit unserer Resultate.

Abstract

Non-smooth saddle-node bifurcations of one-parameter families of quasiperiodically driven dynamical systems on the real line give rise to strange non-chaotic attractors.

In this thesis, we provide a class of families which has non-empty interior in the C^2-topology and whose elements undergo non-smooth saddle-node bifurcations. Within this class, we study the geometry of the corresponding strange attractors by computing different fractal dimensions. In particular, we show that the Hausdorff dimension differs from the box-counting dimension. We further obtain a description of the minimal set at the bifurcation as a maximal invariant set.

Our results treat both the discrete and the continuous time case. A number of explicit examples emphasise the applicability of our findings.

Contents

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1. Introduction

Time and space are the two entities we are used to live, sense, and therefore think in. Against this background, it seems plausible why a mathematical theory which incorporates not just the idea of space but also of time is particularly fascinating and powerful. Of course, the concept of time is visible in several branches of mathematics such as, for example, differential equations. Notwithstanding, the conceptual study of evolution and successive change is reserved for–or in other words: defining–the field of dynamical systems.

In its simplest form, a dynamical system may be given by a pair of a non-empty set Θ, the *phase space*, and a self-mapping f on Θ. What makes up the spirit of dynamics, however, are the kind of questions we ask. At the core of these questions lie objects called *orbits*: given a point $\theta \in \Theta$, its orbit is $\mathcal{O}(\theta) = \{f^n(\theta) : n \in \mathbb{N}_0\}$. Even though the orbit is a rather explicitly provided quantity, it is obvious that it is not just not possible but also not desirable to compute and study every single orbit. Instead, dynamicists search for a description and classification of the *qualitative* behaviour of orbits in order to circumvent the impossibility of considering every orbit separately.

As a first and very important class of orbits, let us consider *fixed points*: a point $\theta \in \Theta$ is called a fixed point if its orbit is a singleton, that is, $f(\theta) = \theta$. Once we are given a fixed point, we are given a lever to open the door to the dynamical investigation of a whole neighbourhood of the fixed point–the reader may from now on think of a phase space equipped with some topology and assume f to be continuous, that is, we are in the realm of topological dynamics. The fixed point might be an attractor: points in its neighbourhood approach it. It might be a repeller: under inversion of time, the fixed point becomes an attractor. Or neither of it: the point might be a saddle.

Another fundamental object which is a generalisation of fixed points are invariant sets: subsets $A \subseteq \Theta$ such that $f(A) = A$. Somewhat closer to the idea of fixed points are minimal invariant sets. These are closed invariant sets which do not contain another (that is, distinct) closed invariant subset. Understanding the decomposition of the phase space into minimal sets, and understanding the dynamics within these minimal sets, is a big part of what topological dynamics is about.

To become more specific, let us study a very simple class of dynamical systems: strictly concave, monotonously increasing C^2-functions $f : \mathbb{R} \to \mathbb{R}$. Our first observation is that the monotonicity implies that two orbits never cross. Secondly, we see that the concavity allows for at most two fixed points $x_1 < x_2$: assuming the existence of a third fixed point x_3 (say, larger than x_2) leads to a contradiction

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} = \frac{x_1 - x_2}{x_1 - x_2} = 1 > \frac{f(x_2) - f(x_3)}{x_2 - x_3} = 1,$$

where the inequality results from the assumption of strict concavity. Thirdly, we see that if we are given two fixed points $x_1 < x_2$, then there exists $x_0 \in (x_1, x_2)$ with $f'(x_0) = 1$. Again due to the strict concavity, this implies that $f' \restriction_{(-\infty,x_0)} > 1$ and $f' \restriction_{(x_0,\infty)} < 1$. This, however, yields that $|f(x) - x_1| = |f(x) - f(x_1)| > |x - x_1|$ if $x_0 \geq x \neq x_1$ and likewise $|f(x) - x_2| < |x - x_2|$ if $x_0 \leq x \neq x_2$. Altogether, this shows: x_1 is a repeller and x_2 an attractor and we have a rather comprehensive description of the dynamics of strictly concave increasing maps on the real line.

Once the dynamics of a given system are understood, it is natural to ask how the dynamical character changes when we change the system. On the one hand, we could consider arbitrary but small (in whatever sense) perturbations of the system and ask to what degree the dynamics of the perturbed systems are comparable to the dynamics of the unperturbed one. This sort of questions concern the so-called *robustness* of a dynamical system. On the other hand, we may consider changes in certain directions in order to produce interesting dynamics by carefully adjusting just a few or only one parameter of the system. This is what bifurcation theory is about: it investigates qualitative change in the long-term behaviour of a dynamical system along a continuous variation of some parameters of the system.

In our above example, we could, for instance, add some parameter dependence to the map f by considering the family $(f_\beta)_{\beta \geq 0}$ with $f_\beta = f - \beta$. What happens along the growth of β? Assuming two fixed points for $f = f_0$, an increase in β results in these two points to approach each other. If β is too big, there are no fixed points anymore. But for a particular parameter, say β_c, we have exactly one fixed point, which is furthermore a saddle. Thus, the dynamics of f_{β_c} are rather different from those of f. This sort of bifurcation scenario goes under the name of *saddle-node bifurcation*.

Certainly, we could have also provided a concave and increasing map like f_{β_c} with just one fixed point by hand. However, it is obvious that it is easier to first construct a function with two fixed points and then drag it down to ensure that at *some* parameter there is only one fixed point left.

In this thesis, we deal with *quasi-periodically forced (qpf) monotone maps*. More precisely, we consider mappings

$$f \colon \mathbb{T}^d \times \mathbb{R} \to \mathbb{T}^d \times \mathbb{R}, \quad (\theta, x) \mapsto (\theta + \omega, \tilde{f}(\theta, x)),$$

where $\tilde{f}(\theta, \cdot)$ is strictly concave and monotonously increasing for all θ, and ω is totally irrational. In other words, there is no fixed point for the rotation $\theta \mapsto \theta + \omega$ and thus there is no fixed point for the map f. Hence, our systems ask for other objects to describe the dynamics qualitatively. A natural replacement turn out to be *invariant graphs*, that is, functions $\phi \colon \mathbb{T}^d \to \mathbb{R}$ whose graphs are invariant. Like fixed points of monotone functions on $\mathbb{R}$, these are barriers which cannot be crossed by an orbit.

It is our goal to construct and study qpf monotone maps that–despite their simple non-chaotic (in the sense of zero entropy) character–allow for interesting invariant graphs and minimal sets. As above, we are aiming at this goal by carefully changing some parameter of our initial system such that it undergoes a bifurcation of saddle-node type. We start with a map with two invariant graphs (an attracting and a repelling one) which approach each other along the growth of some parameter until a threshold is reached beyond which these two invariant graphs have vanished. In contrast to the case of simple functions on the real line, the qpf case allows for a dichotomy at the threshold: either, there is just one invariant graph of the saddle type. Or, there are two invariant graphs–an attracting and a repelling one–which coincide on a residual set and still almost surely differ from each other. The latter case is referred to as a *non-smooth saddle-node bifurcation*. The respective attracting graph is called a *strange non-chaotic attractor* (SNA); the repelling one a *strange non-chaotic repeller* (SNR).

The main achievements of this work are:

- General conditions for the existence of SNA's and thus for the occurrence of non-smooth saddle-node bifurcations. In fact, we will extend previous results and thereby prove that non-smooth saddle-node bifurcations are C^2-generic within families of qpf monotone maps.
- A detailed topological and geometrical description of the respective SNA's, the minimal sets and invariant measures at the bifurcation.
- The extension of the above results to forced one-dimensional differential equations.

Apart from abstract theoretical statements, we provide simple explicit examples of families both of maps and differential equations which fall into the regime of our results.

We would like to mention that the main material of this thesis is the content of a series of three articles. The C^2-genericity of non-smooth saddle-node bifurcations of families of maps is treated in [Fuh14]. The study of the geometrical and topological properties of the SNA's is carried out in a collaboration with Maik Gröger and Tobias Oertel-Jäger [FGJ14]. Finally, the case of families of forced differential equations is dealt with in [Fuh15].

1.1. Notation and terminology

By $\mathbb{N}$ we denote the positive integers, by $\mathbb{N}_0$ the non-negative integers. The cardinality of a set A is denoted by $\#A$.

Given a topological space Θ and a subset $A \subseteq \Theta$, we denote the topological closure of A by $\overline{A}$, the interior of A is denoted by $\operatorname{int} A$ and its boundary $\overline{A} \setminus \operatorname{int} A$ by ∂A. If (Θ, d) is a metric space, the diameter of a subset $A \subseteq \Theta$ is denoted by $|A|$ and the distance between two subsets $A \subseteq \Theta$ and $B \subseteq \Theta$ is $d(A, B) = \inf_{a \in A,\, b \in B} d(a, b)$. The open ball of radius $\varepsilon > 0$ centred at some $\theta \in \Theta$ is denoted by $B_\varepsilon(\theta)$.

Throughout this thesis, we deal with continuous maps defined on a Cartesian product

$$f\colon \Theta\times X\to\Theta\times X,\qquad (\theta,x)\mapsto(\omega(\theta),\tilde f(\theta,x)),\tag{1.1.1}$$

where $X\subseteq\mathbb{R}$ is an interval, Θ is a metrizable space, and $\tilde f\colon\Theta\times X\to X$. The Cartesian product of two topological spaces is always assumed to be endowed with the respective product topology. The crucial point in (1.1.1) is that the first coordinate of the image is independent of the second coordinate of the argument, which is the reason for calling such maps *skew products*. For $\theta\in\Theta$, the map $x\mapsto\tilde f(\theta,x)$ is called a *fibre map*. It is customary to write $f_\theta(\cdot)=\tilde f(\theta,\cdot)$. The fibre map of f^n at θ is denoted by f_θ^n, where $n\in\mathbb{N}_0$. Hence for $n>1$, we have

$$f_\theta^n(x)=\pi_X\circ f^n(\theta,x)=f_{\theta+(n-1)\omega}\circ\ldots\circ f_\theta(x),\tag{1.1.2}$$

where π_X is the canonical projection to the second coordinate. Likewise, we denote by π_Θ the projection to the first coordinate. Note that even though we do not assume global invertibility in (1.1.1), we may write $f^{-1}(\theta,x)$ occasionally when local invertibility is guaranteed. In this sense, we extend the notation $f_\theta^n(x)=\pi_X\circ f^n(\theta,x)$ to negative n. In particular, note that $f_\theta^{-1}(x)=(f_{\theta-\omega})^{-1}(x)$.

In the preliminary section, it suffices to consider Θ to be just a metrizable space, that is, we are a priori not given a differentiable structure on Θ. In this case, a skew product f of the form (1.1.1) is understood to be C^r–in words: *r-times continuously differentiable*– if $\Theta\times X\ni(\theta,x)\mapsto f_\theta^{(s)}(x)$ exists and is continuous for $s=0,1,\ldots,r$. We may write $\partial_x^s f_\theta(x)$ instead of $f_\theta^{(s)}(x)$.

Our results, however, assume more control over the dynamics on Θ. For that reason, from Chapter 3 on, we will consider Θ to be the d-dimensional torus $\mathbb{T}^d=\mathbb{R}^d/\mathbb{Z}^d$. In this setting, we have a natural differentiable structure on $\Theta=\mathbb{T}^d$. Therefore, from Chapter 3 on, we understand f to be C^r only if it is r-times continuously differentiable in the usual sense, that is, as a mapping from $\mathbb{T}^d\times X$ to itself.

We consider $\mathbb{T}^d$ an additive group with the group operation inherited from the usual addition in $\mathbb{R}^d$. With the natural projection from $\mathbb{R}^d$ onto $\mathbb{T}^d$, $\mathbb{R}^d$ is a cover of $\mathbb{T}^d$ (see [KH97, Definition A.1.19]). Quite often, we slightly abuse notation by not distinguishing elements or subsets of the cover $\mathbb{R}^d$ from such of $\mathbb{T}^d$. As an example, we simply write the continuous time rotation with rotation vector $\rho\in\mathbb{R}^d$ as a map $\mathbb{R}\times\mathbb{T}^d\ni(t,\theta)\mapsto\theta+t\cdot\rho$. Analogously, we write $|\theta-\theta'|$ for the distance of $\theta,\theta'\in\mathbb{T}^d$ in the metric inherited from the Euclidean norm $|\cdot|$ in $\mathbb{R}^d$.

We identify the tangent space of $\mathbb{T}^d$ at θ with $\mathbb{R}^d$. When speaking of directional derivatives of $f(\cdot,x)$, we actually have in mind the respective derivatives of a lift of f (see [KH97, Definition A.1.19]). In this sense, given $\vartheta\in\mathbb{R}^d\setminus\{0\}$, it is clear what is to be understood by $\partial_\vartheta f_\theta(x)=\partial_\vartheta\tilde f(\theta,x)$. Higher derivatives are denoted and understood analogously. Typically, we consider ϑ with $|\vartheta|=1$ and write $\vartheta\in\mathbb{S}^{d-1}$ in this case.

If not stated otherwise, when dealing with measures on a topological space, we consider probability measures on the (completed) Borel σ-algebra. Lebesgue measure on $\mathbb{R}^d$ and $\mathbb{T}^d$ is denoted by $\mathrm{Leb}_{\mathbb{R}^d}$ and $\mathrm{Leb}_{\mathbb{T}^d}$, respectively.

2. Preliminaries

This chapter provides the technical background needed to understand the investigation of forced one-dimensional systems carried out in this work.

In the first part, we have a look at the dynamics and the associated invariant objects both from a topological and an ergodic theory point of view. In the last part, we introduce basic notions from fractal geometry which allow us to specify the idea of how "strange" our attractors actually are.

In later chapters, we have to restrict our attention to systems forced by rather specific "driving systems" (see Chapter 3) but for the time being, we can consider a rather wide class of forced one-dimensional systems.

2.1. Forced one-dimensional systems

The first part of this section, that is, everything up to and including Paragraph 2.1.3 treats forced one-dimensional systems from a topological point of view. Despite a more lucid presentation of the proofs and statements, it should be mentioned that most of the–admittedly rather basic–ideas and observations in this part are taken from [Sta03]. The experienced reader might browse or even skip these pages and merely use them as a dictionary when some definition or clarification is needed. However, for the sake of completeness and also to allow the reader to develop an intuition for the objects we encounter in the rest of this work, we consider this part a simple but nicely motivating starting point.

Throughout this thesis, we deal with classes of *driven*–in the following, we may synonymously use the term *forced*–one-dimensional dynamical systems. Their dynamics are defined on the Cartesian product $\Theta \times X$ of a metrizable space Θ, the *base (space)* on which a *driving system* acts, and a possibly non-compact interval $X \subseteq \mathbb{R}$ whose dynamics are coupled to the driving system. More precisely, we consider systems of either of the two following forms.

On the one hand–we refer to this situation as the *discrete time case*–we study certain continuous skew product maps, so-called *forced monotone interval maps*

$$f\colon \Theta \times X \to \Theta \times X, \qquad (\theta, x) \mapsto (\omega(\theta), f_\theta(x)), \tag{2.1.1}$$

where the *base map* $\omega\colon \Theta \to \Theta$ is a homeomorphism and the fibre maps $f_\theta\colon X \to X$ are non-decreasing for each $\theta \in \Theta$.

On the other hand–we refer to this situation as the *continuous time case*–given a *non-autonomous vector field* $F\colon \Theta \times X \to \mathbb{R}$, we study *(local) skew product flows* or, more precisely, *forced one-dimensional (local) flows* of the form

$$\Xi\colon U \subseteq \mathbb{R}\times\Theta\times X \to \Theta\times X, \qquad (t,\theta,x)\mapsto(\rho_t(\theta),\xi(t,\theta,x)), \tag{2.1.2}$$

where $\rho\colon \mathbb{R}\times\Theta \ni (t,\theta)\mapsto\rho_t(\theta)\in\Theta$ is a *flow* (cf. A.1), ξ is the unique (under some mild assumptions on F) maximal solution of

$$\partial_t\xi(t,\theta,x) = F\left(\rho_t(\theta),\xi(t,\theta,x)\right) \tag{2.1.3}$$

with $\xi(0,\theta,x) = x$ for each $(\theta,x)\in\Theta\times X$, and U is the domain of ξ. We refer to equation (2.1.3) as a *differential equation driven by* ρ and likewise call Ξ a *(local) flow driven by* ρ. Given ρ, we may further say Ξ *is generated by* F.

In the following, we assume F to be C^1 if not stated otherwise. Due to the Picard-Lindelöf Theorem, we therefore know that for all $(\theta,x)\in\Theta\times X$ there is a maximal, non-degenerate time-interval $U_{\theta,x}$ containing 0 on which we can take existence and uniqueness of $\xi(\cdot,\theta,x)$ for granted (see, e.g. [Har64, Chapter II, Theorem 1.1 & 3.1]). Note that

$$\xi(t+\tau,\theta,x) = \xi(t,\rho_\tau(\theta),\xi(\tau,\theta,x)) \tag{2.1.4}$$

for $t\in U_{\rho_\tau(\theta),\xi(\tau,\theta,x)} = U_{\theta,x}-\tau$ and $\tau\in U_{\theta,x}$. As further the dependence of ξ on θ and x is continuous (cf. [Har64, Chapter V, Theorem 2.1]), we get that Ξ is indeed a local flow in the sense of Definition A.1.4.

Finally and for further reference, observe that

$$\xi^-\colon (t,\theta,x)\mapsto\xi(-t,\theta,x) \tag{2.1.5}$$

solves (2.1.3) with the right-hand side replaced by

$$F^-\left(\rho_t^-(\theta),\xi^-(t,\theta,x)\right), \tag{2.1.6}$$

where $\rho^-_{(\cdot)} = \rho_{-(\cdot)}$ and $F^- = -F$. Note that $\xi^-(t,\rho_t(\theta),\xi(t,\theta,x)) = \xi(0,\theta,x) = x$ for all $t\in U_{\theta,x}$ due to (2.1.4).

2.1.1. Topology of closed sets

We want to study closed invariant sets of forced systems of the form (2.1.1) and (2.1.2). To that end, let us draw some basic conclusions on the topology of closed subsets in $\Theta\times X$.

Definition 2.1.1. Let $A\subseteq\Theta\times X$. We say A is Θ-*covering* if

$$\pi_\Theta(A) = \Theta,$$

where $\pi_\Theta\colon \Theta \times X \to \Theta$ is the canonical projection to the first coordinate. We call A *bounded* if

$$\inf \bigcup_{\theta\in\Theta} A(\theta),\ \sup \bigcup_{\theta\in\Theta} A(\theta) \in X,$$

where $A(\theta) = \{x \in X\colon (\theta, x) \in A\}$. If A is bounded, closed and Θ-covering, its *upper* and *lower boundary graphs* are functions $\phi_A^+\colon \Theta \to X \subseteq \mathbb{R}$ and $\phi_A^-\colon \Theta \to X \subseteq \mathbb{R}$, respectively, with

$$\phi_A^+\colon \theta \mapsto \sup A(\theta), \qquad \phi_A^-\colon \theta \mapsto \inf A(\theta).$$

Given a function $\phi\colon \Theta \to \mathbb{R}$, we refer by its *graph* to the point set $\{(\theta, \phi(\theta))\colon \theta \in \Theta\} \subseteq \Theta \times X$, denoted by the respective capital letter Φ. Note that in the case of the upper and lower boundary graphs from above (and the invariant graphs introduced in the next section) this introduces a slight and harmless abuse of terminology, since we use the notion *graph* both for the function and the respective point set.

Observe that if A is closed, then $A(\theta)$ is closed for each $\theta \in \Theta$, and thus $\Phi_A^+, \Phi_A^- \subseteq A$. In the following, we denote the set of all neighbourhoods of $\theta \in \Theta$ by $\mathscr{U}_\theta$.

Definition 2.1.2. A function $\phi\colon \Theta \to \mathbb{R}$ is *upper semi-continuous (usc) in* θ if

$$\forall \varepsilon > 0\, \exists U \in \mathscr{U}_\theta\, \forall \theta' \in U : \phi(\theta') < \phi(\theta) + \varepsilon.$$

A function $\phi\colon \Theta \to \mathbb{R}$ is *lower semi-continuous (lsc) in* θ if $-\phi$ is upper semi-continuous in θ. If ϕ is upper (lower) semi-continuous for all $\theta \in \Theta$, we say ϕ is *upper (lower) semi-continuous*.

Remark. If ϕ is usc (lsc), then $a \cdot \phi$ is usc (lsc), while $-a \cdot \phi$ is lsc (usc) for $a > 0$. The sum of two usc (lsc) functions is usc (lsc), too.

We omit the proof of the next, rather well-known fact.

Lemma 2.1.3 (cf. [Sta03, Lemma 3])**.** *A function $\phi\colon \Theta \to \mathbb{R}$ is upper semi-continuous if and only if*

$$\{(\theta, x) \in \Theta \times \mathbb{R}\colon x \leq \phi(\theta)\}$$

is closed. Similarly, ϕ is lower semi-continuous if and only if

$$\{(\theta, x) \in \Theta \times \mathbb{R}\colon x \geq \phi(\theta)\}$$

is closed.

Further, if Θ is compact, an usc function realises its supremum and a lsc function realises its infimum.

Corollary 2.1.4 (cf. [Sta03, Corollary 2])**.** *Suppose $A \subseteq \Theta \times X$ is bounded from above (below) by an usc (lsc) function $\phi\colon \Theta \to \mathbb{R}$, that is, for $(\theta, x) \in A$ we have that $\phi(\theta) \geq x$ ($\phi(\theta) \leq x$). Then $\overline{A}$ is also bounded from above (below) by ϕ.*

Proof. If ϕ is usc, then $W = \{(\theta, x) \colon x \leq \phi(\theta)\}$ is closed by Lemma 2.1.3. Since $A \subseteq W$, we have $\overline{A} \subseteq W$. The case with the lower boundary follows similarly. □

This immediately implies the following.

Corollary 2.1.5. *Suppose ϕ is usc. Then $\phi^+_{\overline{\Phi}} = \phi$, that is, the upper boundary graph of $\overline{\Phi}$ coincides with ϕ. Likewise, if ϕ is lsc, then $\phi^-_{\overline{\Phi}} = \phi$.*

Lemma 2.1.6 (cf. [Sta03, Corollary 1])**.** *Let $A \subseteq \Theta \times X$ be a bounded, closed, and Θ-covering set with boundary graphs ϕ^+_A and ϕ^-_A. Then ϕ^+_A is usc and ϕ^-_A is lsc.*

Proof. Given (θ_n, x_n) in $W = \{(\theta, x) \in \Theta \times \mathbb{R} \colon x \leq \phi^+_A(\theta)\}$ with $\lim_{n\to\infty}(\theta_n, x_n) = (\theta, x)$, we may assume without loss of generality that the sequence $(\theta_n, \phi^+_A(\theta_n))$ in $\Phi^+_A \subseteq A$ converges to $(\theta, x') \in A$ for some $x' \in X$ since A is closed and bounded. Note that $x' \leq \phi^+_A(\theta)$. Now, $x_n \leq \phi^+_A(\theta_n)$ so that $x \leq x' \leq \phi^+_A(\theta)$. Thus, W is closed and ϕ^+_A is usc by Lemma 2.1.3. The case with ϕ^-_A follows similarly. □

2.1.2. Invariant sets and invariant graphs

Now, we consider closed sets which are invariant under the dynamics of either a skew product map f of the form (2.1.1) or a skew product flow Ξ of the form (2.1.2) with a non-autonomous vector field F.

Definition 2.1.7. A set $A \subseteq \Theta \times X$ is *invariant* under f if

$$f(A) = A. \tag{2.1.7}$$

A set $A \subseteq \Theta \times X$ is *invariant* under Ξ if $\mathbb{R} \times A \subseteq U$ and

$$\Xi(t, A) = A, \tag{2.1.8}$$

for all $t \in \mathbb{R}$.

Definition 2.1.8. An *invariant graph of f* is a measurable function $\phi \colon \Theta \to X$ whose graph $\Phi = \{(\theta, \phi(\theta)) \colon \theta \in \Theta\}$ is invariant under f, or equivalently,

$$f_\theta(\phi(\theta)) = \phi(\omega(\theta)),$$

for all $\theta \in \Theta$. An *invariant graph of* Ξ–we might occasionally speak of invariant graphs of the non-autonomous vector field F if the base flow is clear from the context–is a measurable function $\psi \colon \Theta \to X$ whose graph $\Psi = \{(\theta, \psi(\theta)) \colon \theta \in \Theta\}$ is invariant under Ξ, or equivalently,

$$\xi(t, \theta, \psi(\theta)) = \psi(\rho_t(\theta)),$$

for all $\theta \in \Theta$ and all $t \in \mathbb{R}$.

For the sake of readability, we only formulate statements in their discrete time version for the rest of this chapter. However, if not stated otherwise, every such statement as well as all proofs remain essentially the same when the continuous time case is considered.

We call a function $\phi\colon \Theta \to X$ *bounded* if its graph is bounded as a subset of $\Theta \times X$ in the sense of Definition 2.1.1

Lemma 2.1.9 (cf. [Sta03, Lemma 2])**.** *Suppose $\phi\colon \Theta \to X$ is a bounded usc (lsc) invariant graph. Then $\overline{\Phi} \subseteq \Theta \times X$ is a bounded and invariant Θ-covering set whose upper (lower) boundary graph equals ϕ.*

Proof. We only have to show invariance, the rest is either trivial or follows from Corollary 2.1.5.

Note that continuity of f and invariance of Φ give $f\left(\overline{\Phi}\right) \subseteq \overline{f(\Phi)} = \overline{\Phi}$. To see $\overline{\Phi} \subseteq f\left(\overline{\Phi}\right)$, take $(\theta, x) \in \overline{\Phi}$ and choose θ_n so that $\lim_{n\to\infty}(\theta_n, \phi(\theta_n)) = (\theta, x)$. As Φ is bounded, we may assume without loss of generality that $(\omega^{-1}(\theta_n), \phi(\omega^{-1}(\theta_n)))$ converges to $(\omega^{-1}(\theta), x') \in \overline{\Phi}$ for some $x' \in X$. Now, $f(\omega^{-1}(\theta), x') = (\theta, x)$. □

We straightforwardly get the following converse.

Lemma 2.1.10 (cf. [Sta03, Lemma 1])**.** *Let $A \subseteq \Theta \times X$ be a bounded, closed, and invariant Θ-covering set for f. Then the boundary graphs ϕ_A^+ and ϕ_A^- are upper and lower semi-continuous invariant graphs, respectively.*

Proof. The semi-continuity follows from Lemma 2.1.6. As A is invariant and ω is injective, we have $f_\theta(A(\theta)) = A(\omega(\theta))$. By monotonicity of f_θ, we conclude that $f_\theta(\phi_A^+(\theta)) \geq f_\theta(x)$ for any $x \in A(\theta)$ and hence $f_\theta(\phi_A^+(\theta)) \geq y$ for each $y \in A(\omega(\theta))$; in other words: $f_\theta(\phi_A^+(\theta)) = \phi_A^+(\omega(\theta))$. ϕ_A^- can be treated similarly. □

Definition 2.1.11. Let $A \subseteq \Theta \times X$ be a bounded, closed, and invariant Θ-covering set. The *filled-in set* $\mathfrak{Fill}(A)$ of A is

$$\mathfrak{Fill}(A) = \{(\theta, x) \in \Theta \times X\colon \phi_A^-(\theta) \leq x \leq \phi_A^+(\theta)\}.$$

Proposition 2.1.12 (cf. end of Section 2 in [Sta03])**.** *The filled-in set $\mathfrak{Fill}(A) \subseteq \Theta \times X$ of a bounded, closed, and invariant Θ-covering set A is a bounded, closed, and invariant Θ-covering set, too.*

Proof. It is straightforward to see that $\mathfrak{Fill}(A)$ is bounded, invariant and Θ-covering. Further, note that $\mathfrak{Fill}(A)$ equals

$$\{(\theta, x) \in \Theta \times \mathbb{R}\colon \theta \in \Theta \text{ and } x \leq \phi_A^+(\theta)\} \cap \{(\theta, x) \in \Theta \times \mathbb{R}\colon \theta \in \Theta \text{ and } x \geq \phi_A^-(\theta)\}$$

which is closed due to Lemma 2.1.3, since ϕ_A^+ and ϕ_A^- are usc and lsc, respectively (see Lemma 2.1.6). □

From Corollary 2.1.4, Lemma 2.1.9, and Proposition 2.1.12, we get the following.

Corollary 2.1.13. *Suppose $\phi^+ \geq \phi^-$ are bounded invariant graphs with ϕ^+ usc and ϕ^- lsc. Then $[\phi^-, \phi^+] = \{(\theta, x)\colon x \in [\phi^-(\theta), \phi^+(\theta)]\} = \mathfrak{Fill}\left(\overline{\Phi^-} \cup \overline{\Phi^+}\right)$ is a bounded, closed, and invariant Θ-covering set.*

2.1.3. Ordering properties of invariant graphs

From now on, we restrict our attention to minimal base maps ω on a compact and metrizable base Θ. Note that in this case closed and bounded invariant sets of (2.1.1) are in fact compact and necessarily Θ-covering.

Lemma 2.1.14 (cf. [Sta03, Corollary 3])**.** *Let $\phi^+, \phi^- \colon \Theta \to X$ be invariant graphs with ϕ^+ usc and ϕ^- lsc. If $\phi^+(\theta_0) \geq \phi^-(\theta_0)$ for some $\theta_0 \in \Theta$, then $\phi^+(\theta) \geq \phi^-(\theta)$ for each $\theta \in \Theta$.*

Proof. Monotonicity of the fibre maps gives $\phi^+(\theta) \geq \phi^-(\theta)$ for all $\theta \in \mathcal{O}^+(\theta_0) = \{\omega^n(\theta_0)\colon n \in \mathbb{N}\}$. Let $\theta_1 \in \Theta$. By upper and lower semi-continuity of ϕ^+ and ϕ^-, respectively, as well as by minimality of ω, we have that for each $\varepsilon > 0$ there is $U \in \mathscr{U}_{\theta_1}$ and $\theta \in U \cap \mathcal{O}^+(\theta_0)$ such that $\phi^+(\theta_1) + \varepsilon > \phi^+(\theta) \geq \phi^-(\theta) > \phi^-(\theta_1) - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude $\phi^+(\theta_1) \geq \phi^-(\theta_1)$. □

Corollary 2.1.15 (cf. [Sta03, Lemma 4])**.** *Let $\phi^-, \phi^+ \colon \Theta \to X$ be invariant graphs and suppose ϕ^- is usc and ϕ^+ is lsc. If $\phi^-(\theta_0) < \phi^+(\theta_0)$ for some $\theta_0 \in \Theta$, then there exists $\varepsilon > 0$ such that $\phi^-(\theta) < \phi^+(\theta) + \varepsilon$ for each $\theta \in \Theta$.*

Proof. By Lemma 2.1.14, we necessarily have $\phi^+(\theta) - \phi^-(\theta) > 0$ for each $\theta \in \Theta$. Since Θ is compact, the lsc function $\phi^+ - \phi^-$ realises its infimum (Lemma 2.1.3) which is hence larger then 0. □

Definition 2.1.16. Given two invariant graphs ϕ^+ and ϕ^-, we call a point $\theta_0 \in \Theta$ *pinched (with respect to ϕ^+ and ϕ^-)* if $\phi^+(\theta_0) = \phi^-(\theta_0)$.

Lemma 2.1.17 (cf. [Sta03, Lemma 5])**.** *Suppose ϕ^+ is an usc and ϕ^- is a lsc invariant graph and there is a pinched point $\theta_0 \in \Theta$. Then*

(a) The set of pinched points is residual in Θ.

(b) ϕ^+ and ϕ^- are simultaneously continuous in $\theta \in \Theta$ if and only if θ is pinched.

Proof. First of all, note that $\phi^+ \geq \phi^-$ by Lemma 2.1.14. Set $A_n = \{\theta \in \Theta\colon \phi^+(\theta) - \phi^-(\theta) < 1/n\}$. Since $\phi^+ - \phi^-$ is usc, A_n is open and since it contains $\mathcal{O}^+(\theta_0)$, A_n is further dense. Thus, $\{\theta \in \Theta\colon \theta \text{ is pinched}\} = \bigcap_{n\in\mathbb{N}} A_n$ is residual.

Now, if θ_0 is pinched, then for each $\varepsilon > 0$ there is a neighbourhood $U \in \mathscr{U}_{\theta_0}$ such that for each $\theta \in U$ we have $\phi^-(\theta_0)+\varepsilon = \phi^+(\theta_0)+\varepsilon > \phi^+(\theta) \geq \phi^-(\theta) > \phi^-(\theta_0)-\varepsilon = \phi^+(\theta_0)-\varepsilon$ where we used the semi-continuity of ϕ^+ and ϕ^-. This proves continuity of ϕ^+ and ϕ^- in the pinched points. Further, given a point of continuity θ_0 for both ϕ^+ and ϕ^-, we observe that if we had $\phi^+(\theta_0) > \phi^-(\theta_0)$, then $\phi^+ > \phi^-$ was true in a whole neighbourhood of θ_0 contradicting the fact that the pinched points are dense. □

Definition 2.1.18. A bounded and closed invariant set A is called *pinched* if there is $\theta_0 \in \Theta$ which is pinched with respect to the boundary graphs ϕ_A^+ and ϕ_A^-. A is called *thin* if either $\Phi_A^+ \subseteq \overline{\Phi_A^-}$ or $\Phi_A^- \subseteq \overline{\Phi_A^+}$.

The following two statements are straightforward.

Lemma 2.1.19 (cf. [Sta03, Lemma 6]). *Every pinched set A contains exactly one minimal set.*

Proof. The existence of a minimal set is a well-known general implication of Zorn's Lemma (see Proposition A.1.6). The uniqueness follows from the fact that the minimal set has to contain the orbit of $(\theta_0, \phi_A^+(\theta_0)) = (\theta_0, \phi_A^-(\theta_0))$, where θ_0 is a pinched point. □

Lemma 2.1.20 (cf. [Sta03, Corollary 4]). *Every minimal set A is thin. Further, $A = \overline{\Phi_A^-} = \overline{\Phi_A^+}$.*

Proof. Minimality implies that every orbit is dense. Further, both Φ_A^+ and Φ_A^- contain entire orbits. □

2.1.4. Invariant graphs from a measure-theoretic perspective

In the previous paragraphs, we restricted ourselves essentially to a topological point of view on invariant graphs. Now, we assume a measure theoretic perspective. For simplicity, let us suppose from now on–without further mentioning–that the driving system is uniquely ergodic, that is, it has a unique invariant measure m, which is hence ergodic (see Proposition A.2.4).

Given an invariant graph ϕ of (2.1.1), the map $g\colon \Theta \ni \theta \mapsto (\theta, \phi(\theta)) \in \Theta \times X$ is measurable. Hence, if $A \in \Theta \times X$ is a Borel set, then $\pi_\Theta(A \cap \Phi) = g^{-1}(A)$ is a Borel set in Θ, too, so that $\mu_\phi\colon A \mapsto m(\pi_\Theta(A \cap \Phi))$ is well-defined. In fact, it is easy to see that μ_ϕ is a measure on $\Theta \times X$ which is further invariant, since $\pi_\Theta(f^{-1}(A) \cap \Phi) = \pi_\Theta(f^{-1}(A \cap \Phi)) = \omega^{-1}(\pi_\Theta(A \cap \Phi))$. Moreover, note that for A with $f^{-1}(A) = A$, we have $\pi_\Theta(A \cap \Phi) = \omega^{-1}(\pi_\Theta(A \cap \Phi))$ so that $m(\pi_\theta(A \cap \Phi))$ is either 1 or 0, due to the ergodicity of m. We have thus shown that to each invariant graph ϕ we can associate an ergodic f-invariant measure μ_ϕ.

The following theorem provides us with a converse of this observation and thereby underlines the importance of invariant graphs in the study of (2.1.1) from an ergodic theory point of view. We call a measurable function $\phi\colon \Theta \to X$ an *(f, m)-invariant graph* if

$$f_\theta(\phi(\theta)) = \phi(\omega(\theta)) \tag{2.1.9}$$

m-almost surely.

Theorem 2.1.21 (cf. [Für61, Theorem 4.1] & [Arn98, Theorem 1.8.4]). *Let f be a forced monotone interval map. To every ergodic f-invariant measure μ there is an (f, m)-invariant graph ϕ with $\mu = \mu_\phi$.*

Remark. Given an (f, m)-invariant graph ϕ, set

$$N = \{\omega^n(\theta) \colon n \in \mathbb{Z} \text{ and } \theta \in \Theta \text{ is such that (2.1.9) is not verified}\}.$$

If additionally, there is an f-invariant graph ϕ_0 in the sense of Definition 2.1.8, we may define

$$\tilde{\phi}(\theta) = \begin{cases} \phi(\theta) & \text{if } \theta \notin N \\ \phi_0(\theta) & \text{if } \theta \in N. \end{cases}$$

Now, $\tilde{\phi}$ is an invariant graph[1] which coincides almost surely with ϕ. In the following, we will apply the above theorem only in such situations in which we are a priori given a compact invariant set whose boundary graphs can hence play the part of ϕ_0 (cf. Lemma 2.1.10). In this sense, we may assume without loss of generality that to each ergodic f-invariant measure μ there is an f-invariant graph ϕ with $\mu = \mu_\phi$.

In the following, we assume f to be C^1 if not stated otherwise. Note that in the continuous time case, ξ is C^1 with respect to x anyway since we assume F to be C^1 (see, e.g. [Har64, Chapter V, Corollary 3.1]). To decide whether the orbit of a point $(\theta, x) \in \Theta \times X$ attracts or repels nearby points on the same fibre under the iteration of f and Ξ, it is natural to study the *forward (vertical) Lyapunov exponent* $\lambda^+(\theta, x)$

$$\lambda^+(\theta, x) = \lim_{n\to\infty} 1/n \cdot \log |\partial_x f_\theta^n(x)| \quad \text{and} \quad \lambda^+(\theta, x) = \lim_{t\to\infty} 1/t \cdot \log |\partial_x \xi(t, \theta, x)|,$$

respectively if the corresponding limits exist. Similarly, to characterise the dynamics close to f- and Ξ-invariant graphs ϕ and ψ, respectively, we introduce the *Lyapunov exponent*

$$\lambda(\phi) = \int_\Theta \log |\partial_x f_\theta(\phi(\theta))| \; dm(\theta) \quad \text{and} \quad \lambda(\psi) = 1/t \cdot \int_\Theta \log |\partial_x \xi\,(t, \theta, \psi(\theta))| \; dm(\theta),$$

where the latter is easily seen to be independent of the particular choice of $t > 0$. In the following, we again restrict our discussion to the discrete time case.

Suppose we are given an invariant graph ϕ. By the chain rule and Birkhoff's Ergodic Theorem (Theorem A.2.5), we have for almost every $\theta \in \Theta$ that

$$\begin{aligned} \lambda^+(\theta, \phi(\theta)) &= \lim_{n\to\infty} 1/n \cdot \log |\partial_x f_\theta^n(\phi(\theta))| = \lim_{n\to\infty} 1/n \cdot \sum_{\ell=0}^{n-1} \log |\partial_x f_{\theta+\ell\omega}(\phi(\theta + \ell\omega))| \\ &= \lambda(\phi). \end{aligned} \tag{2.1.10}$$

For simplicity, let us assume for now that $\partial_x f_\theta(\phi(\theta)) > 0$ for all $\theta \in \Theta$. If ϕ is continuous, $f\restriction_\Phi \colon \Phi \to \Phi$ is a uniquely ergodic topological dynamical system on a compact space so that by the uniform ergodic theorem (cf. Theorem A.2.6), we have equation (2.1.10) for

[1] In order to see the measurability, recall that we consider the completed Borel σ-algebra on Θ.

every $\theta \in \Theta$. Assume we are given a continuous invariant graph ϕ with $\lambda = \lambda(\phi) < 0$. By the uniform ergodic theorem, there is $N \in \mathbb{N}$ such that $\frac{1}{n} \log |\partial_x f_\theta^n(\phi(\theta))| < \lambda/2$ for all $n \geq N$ and all $\theta \in \Theta$. By compactness of Θ, there is hence $\delta > 0$ such that $[\phi - \delta, \phi + \delta]$ gets mapped into $[\phi - \delta \cdot \exp(\ell\lambda/3), \phi + \delta \cdot \exp(\ell\lambda/3)]$ under ℓN iterations of f. We have thus a uniform convergence towards Φ in a whole neighbourhood of it. We therefore say an invariant graph ϕ is an *attractor* if $\lambda(\phi) < 0$; we call it *repeller* if $\lambda(\phi) > 0$; and we call it *neutral* if $\lambda(\phi) = 0$.

In general, we do not have a whole strip around an attractor Φ in which convergence to Φ occurs, instead the next statement–which follows from Pesin theory if f is a $C^{1+\alpha}$-diffeomorphism on a smooth manifold (cf. supplement in [KH97])–holds.

Theorem 2.1.22 ([Jäg03, Proposition 3.3])**.** *Suppose ϕ is an invariant graph with $\lambda(\phi) < 0$. Then there is an almost surely positive function $\delta\colon \Theta \to \mathbb{R}$ such that for all $(\theta, x) \in [\phi - \delta, \phi + \delta]$ we have*

$$|f^n(\theta, x) - f^n(\theta, \phi(\theta))| \overset{n\to\infty}{\longrightarrow} 0.$$

The above discussion carries over to a repeller by sending $n \to -\infty$ and replacing $\lambda^+(\theta, x)$ by the *backward (vertical) Lyapunov exponent* $\lambda^-(\theta, x)$ at a point $(\theta, x) \in \Theta \times X$ given by

$$\lambda^-(\theta, x) = \lim_{n\to\infty} 1/n \cdot \log |\partial_x f_\theta^{-n}(x)|,$$

where we implicitly assume local invertibility of f^n at (θ, x) for all $n \in \mathbb{N}$. The latter is, in particular, guaranteed for points in a compact invariant set A if $\partial_x f_\theta > 0$ on A.

2.1.5. Bifurcations of invariant graphs

Given an unforced (in other words, autonomous) monotone interval map g, strict concavity of g implies the existence of at most two distinct fixed points of which the upper one is attracting, while the lower one is repelling. If g depends continuously and monotonously decreasingly on a parameter $\beta \in [0, 1]$, such two fixed points approach each other along the growth of β until they possibly merge to a single fixed point x_0 which–due to the concavity–turns out to be *neutral*, that is, $g'(x_0) = 1$. This qualitative change of the dynamics from two fixed points (attracting and repelling, respectively), to a single (neutral) one goes under the name of *saddle-node* bifurcation. It is obvious that concavity plays an important part in this bifurcation scenario.

In our non-autonomous setting, we study bifurcations of invariant graphs $(\phi_\beta)_{\beta\in[0,1]}$ of skew product families $(f_\beta)_{\beta\in[0,1]}$. As we are interested in saddle-node bifurcations, we likewise study "collisions" of an attracting with a repelling graph. The next theorem tells us that–under some mild additional technical assumptions–concavity of the fibre maps f_θ is still a crucial ingredient for bifurcations of the saddle-node type in the non-autonomous setting.

Theorem 2.1.23 (cf. [AJ12, Theorem 2.1])**.** *Consider a C^2-map f of the form* (2.1.1). *Assume that for each $\theta \in \Theta$ there exist measurable functions $\gamma^- \leq \gamma^+ \colon \Theta \to X$ such that for all $\theta \in \Theta$ the fibre maps are strictly concave on $\Gamma(\theta) = [\gamma^-(\theta), \gamma^+(\theta)]$. Further, assume that $h(\theta) = \inf_{x \in \Gamma(\theta)} \log f'_\theta(x)$ has an integrable minorant.*

Then there exist at most two distinct invariant graphs (up to equivalence on full-measure sets) in $\Gamma = \{(\theta, x) \in \Theta \times X \colon x \in \Gamma(\theta)\}$. Moreover, if there exist two distinct invariant graphs $\phi^- \leq \phi^+$ in Γ, then ϕ^- is a repeller and ϕ^+ is an attractor.

Remark. From now on, we identify invariant graphs which coincide m-almost surely. We henceforth say an invariant graph verifies a certain property if there is an invariant graph in its equivalence class satisfying this property. In the particular case of the above statement, we hence understand an invariant graph ϕ to be *in* a set $\Gamma \subseteq \Theta \times X$ if $\tilde{\phi}(\theta) \in \Gamma(\theta)$ for some invariant representative $\tilde{\phi}$ of its equivalence class and all $\theta \in \mathbb{T}^d$. Similarly, we say a graph is *continuous* if there is a continuous representative of its equivalence class. Note that thus a *non-continuous* graph is a graph which does not allow for a continuous representative.

Theorem 2.1.23 provides a basis for a setup for saddle-node bifurcations of forced systems. As we study *local* bifurcations, we restrict to a section $\Gamma = \Theta \times [\gamma^-, \gamma^+]$ of the phase space, where–in contrast to the previous statement and for simplicity only–we assume $\gamma^- < \gamma^+ \in \mathbb{R}$ to be constant. In the following, we consider families whose members have the same driving system and the same X with a non-degenerate interval $[\gamma^-, \gamma^+] \subseteq X$.

In order to ensure the occurrence of a saddle-node bifurcation, we need to impose a number of further conditions on the families of systems under investigation. Let us consider the discrete and continuous time case separately.

Given a family $(f_\beta)_{\beta \in [0,1]}$ of C^2-maps of the form (2.1.1), we assume the following for all $\beta \in [0, 1]$ and all $\theta \in \Theta$ if applicable.

$$f_{\beta,\theta}(\gamma^-) \leq \gamma^- \quad \text{and} \quad f_{\beta,\theta}(\gamma^+) \leq \gamma^+; \tag{2.1.11}$$

$$f'_{\beta,\theta}(x) > 0 \quad \text{for all } x \in [\gamma^-, \gamma^+]; \tag{2.1.12}$$

$$f''_{\beta,\theta}(x) < 0 \quad \text{for all } x \in [\gamma^-, \gamma^+]; \tag{2.1.13}$$

$$\partial_\beta f_{\beta,\theta}(x) \leq 0 \text{ and there is } \theta_0 \text{ such that } \partial_\beta f_{\beta,\theta_0}(x) < 0 \text{ for all } x \in [\gamma^-, \gamma^+]; \tag{2.1.14}$$

$$f_0 \text{ has two continuous invariant graphs and } f_1 \text{ has no invariant graph in } \Gamma. \tag{2.1.15}$$

Theorem 2.1.24. (cf. [AJ12, Theorem 6.1]) *Suppose $(f_\beta)_{\beta \in [0,1]}$ is a family of forced monotone C^2-interval maps of the form* (2.1.1) *that depend continuously differentiable on the family parameter*[2] *β where the base Θ, the base map ω, and $X \supseteq [\gamma^-, \gamma^+]$ with fixed $\gamma^- < \gamma^+$ coincide for all β. Further, assume* (2.1.11)–(2.1.15) *are verified. Then there exists a unique* critical parameter $\beta_c \in (0, 1)$ *such that there holds:*

(i) If $\beta < \beta_c$, then there exist exactly two continuous f_β-invariant graphs $\phi^-_\beta < \phi^+_\beta$ in Γ with $\lambda(\phi^-_\beta) > 0$ and $\lambda(\phi^+_\beta) < 0$.

[2] Hereby, we mean that the map $(\beta, \theta, x) \mapsto \partial_\beta f_{\beta,\theta}(x)$ is well-defined and continuous.

(ii) If $\beta > \beta_c$, then there exists no f_β-invariant graph in Γ.

(iii) If $\beta = \beta_c$, then one of the following two possibilities hold.

($\mathcal{S}$) Smooth bifurcation: f_{β_c} has a unique invariant graph ϕ_{β_c} in Γ, which satisfies $\lambda(\phi_{\beta_c}) = 0$. Either ϕ_{β_c} is continuous, or it contains both an upper and lower semi-continuous representative in its equivalence class.

($\mathcal{N}$) Non-smooth bifurcation: f_{β_c} has exactly two invariant graphs $\phi^-_{\beta_c} < \phi^+_{\beta_c}$ a.s. in Γ. The graphs $\phi^-_{\beta_c}$ and $\phi^+_{\beta_c}$ are pinched, $\phi^-_{\beta_c}$ is lower semi-continuous, whereas $\phi^+_{\beta_c}$ is upper semi-continuous, but none of the graphs is continuous. Further, $\lambda(\phi^-_{\beta_c}) > 0$ and $\lambda(\phi^+_{\beta_c}) < 0$.

Remark. (a) In line with expectations, a symmetric version with convex fibre maps and/or $\partial_\beta f_{\beta,\theta}(x) > 0$ holds true as well [AJ12, Remark 4.3 (d)].

(b) As a matter of fact, condition (2.1.14) reads stricter in [AJ12] where the strict inequality is assumed to hold for all $\theta \in \Theta$ and all $x \in [\gamma^-, \gamma^+]$. However, this condition is just to ensure the uniqueness of the critical parameter β_c, which is also guaranteed by our slightly weaker assumption.

(c) Given a family $(f_\beta)_{\beta\in[0,1]}$ of forced monotone C^2-interval maps of the form (2.1.1) which depends continuously on β, we say in the following that it undergoes a *saddle-node bifurcation (smooth or non-smooth)* in $\Gamma = \Theta \times [\gamma^-, \gamma^+]$ if there exists a unique $\beta_c \in (0, 1)$ such that the conclusions of Theorem 2.1.24 hold (even if the assumptions may not be verified).

Coming to saddle-node bifurcations in continuous time, the main difference is that the "evolution law" is not explicitly provided, but implicitly given by a differential equation that we still need to integrate. This problem typically arises when dealing with flows. In order to yield applicable statements, the task is thus to translate desirable properties of the flow (or an appropriate return map on a Poincaré section) in properties of the vector field. This being said, on a dynamical level the situation is–not surprisingly–similar.

Now, given a family $(\Xi_\beta)_{\beta\in[0,1]}$ of flows of the form (2.1.2) with non-autonomous C^2-vector-fields $(F_\beta)_{\beta\in[0,1]}$, we assume the following for all $\beta \in [0, 1]$ and all $\theta \in \Theta$ if applicable.

$$F_\beta(\theta,\gamma^+) \leq 0 \quad \text{and} \quad F_\beta(\theta,\gamma^-) \leq 0; \tag{2.1.16}$$
$$\partial_x^2 F_\beta(\theta, x) < 0 \quad \text{for all } x \in [\gamma^-, \gamma^+]; \tag{2.1.17}$$
$$\partial_\beta F_\beta(\theta, x) \leq 0 \text{ and there is } \theta_0 \text{ such that } \partial_\beta F_\beta(\theta_0, x) < 0 \text{ for all } x \in [\gamma^-, \gamma^+]; \tag{2.1.18}$$
$$F_0 \text{ has two continuous invariant graphs and } F_1 \text{ has no invariant graph in } \Gamma. \tag{2.1.19}$$

For a detailed discussion of how (2.1.11)–(2.1.15) translate to (2.1.16)–(2.1.19), see [AJ12, Section 7]. The reader may further consult Section 6.1.

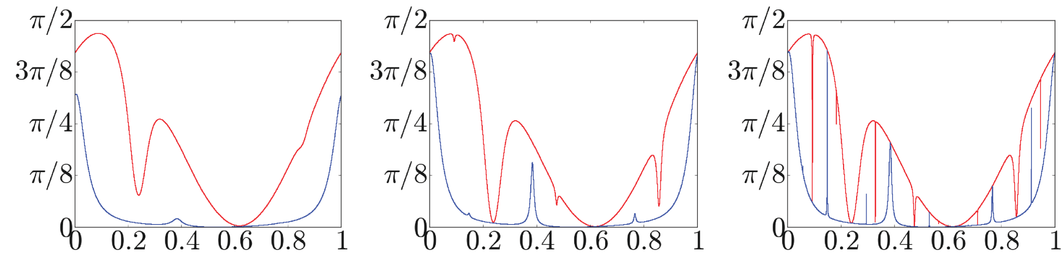

Figure 2.1. The non-smooth saddle-node bifurcation of the family $(\theta, x) \mapsto (\theta + \omega, \arctan(ax) - \beta[1 + \cos 2\pi\theta])$, where ω is the golden mean and $a = 40$. We see how the invariant graphs approach each other on a measure zero set as we increase the parameter β from the left to the right. The red graph is attracting; the blue one repelling. (a) $\beta = 0.77$; (b) $\beta = 0.7729$; (c) $\beta = 0.7729846$.

Theorem 2.1.25 (cf. [NO08, Theorem 3.1], [AJ12, Theorem 7.1])**.** *Consider a C^2-family of non-autonomous vector fields $(F_\beta)_{\beta\in[0,1]}$ depending continuously differentiable on β where the base Θ, the base flow ρ, and $X \supseteq [\gamma^-, \gamma^+]$ with fixed $\gamma^- < \gamma^+$ coincide for all β. Further, assume* (2.1.16)–(2.1.19) *are verified.*

Then there exists a unique critical parameter $\beta_c \in (0, 1)$ *such that there holds:*

(i) If $\beta < \beta_c$, then there exist exactly two continuous invariant graphs $\psi^-_\beta < \psi^+_\beta$ of F_β in Γ with $\lambda(\psi^-_\beta) > 0$ and $\lambda(\psi^+_\beta) < 0$.

(ii) If $\beta > \beta_c$, then there exists no invariant graph of F_β in Γ.

(iii) If $\beta = \beta_c$, then one of the following two possibilities hold.

($\mathcal{S}$) Smooth bifurcation: *F_{β_c} has a unique invariant graph ψ_{β_c} in Γ, which satisfies $\lambda(\psi_{\beta_c}) = 0$. Either ψ_{β_c} is continuous, or it contains both an upper and lower semi-continuous representative in its equivalence class.*

($\mathcal{N}$) Non-smooth bifurcation: *F_{β_c} has exactly two invariant graphs $\psi^-_{\beta_c} < \psi^+_{\beta_c}$ a.s. in Γ. The graphs $\psi^-_{\beta_c}$ and $\psi^+_{\beta_c}$ are pinched, $\psi^-_{\beta_c}$ is lower semi-continuous, whereas $\psi^+_{\beta_c}$ is upper semi-continuous, but none of the graphs is continuous. Further, $\lambda(\psi^-_{\beta_c}) > 0$ and $\lambda(\psi^+_{\beta_c}) < 0$.*

The novelty with respect to the autonomous situation, both in the discrete and the continuous time case, is the dichotomy at the critical parameter. While smooth bifurcations can be easily realised by considering direct products of any base map and suitable interval map families, the existence of non-smooth bifurcations is much more difficult to establish. Hence, it is the alternative of the non-smooth bifurcation we are interested in and the objects arising therein (see Figure 2.1) are the central theme of this thesis.

Definition 2.1.26. A non-continuous invariant graph ϕ is called a *strange non-chaotic attractor (SNA)* if $\lambda(\phi) < 0$; it is called a *strange non-chaotic repeller (SNR)* if $\lambda(\phi) > 0$.

Remark. The notion of "strangeness" is not reserved to the above objects and there is no specific definition of what should be considered a strange attractor in general. Nevertheless, from Figure 2.1(c) we get at least an intuitive idea of what could be regarded as "strange" about SNA's.

In the present context, the notion goes back to an article by Grebogi et al. from 1984, where numerical evidence and heuristic arguments for the existence of an SNA (in the above sense) are found for a rather particular class of forced one-dimensional systems (cf. [GOPY84, Kel96]). However, rigorous results establishing the existence of SNA's (at least implicitly) had already been derived before [Mil68, Vin75, Her83].

In the subsequent chapters, it is our goal to study the existence and properties of the SNA's and SNR's that arise–not necessarily, but particularly–in non-smooth saddle-node bifurcations. We hence need a criterion to judge whether a bifurcation is smooth or non-smooth. An important ingredient of our criterion is the following statement which is a generalisation of the uniform ergodic theorem (cf. Theorem A.2.6). Again, we restrict to the discrete time case.

Theorem 2.1.27 (Semi-uniform Birkhoff ergodic theorem, cf. [SS00, Theorem 1.9])**.** *Suppose we are given a continuous map $T\colon M \to M$ on a compact metrizable space M, a continuous function $h\colon M \to \mathbb{R}$, and $a \in \mathbb{R}$ such that*

$$\int_M h\, d\mu \le a$$

for all T-invariant measures μ. Then given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \ge N$ and all $z \in M$

$$\frac{1}{n}\sum_{\ell=0}^{n-1} h\left(T^{\ell}(z)\right) \le a + \varepsilon. \tag{2.1.20}$$

Corollary 2.1.28. *Consider a compact invariant set A with $f'_\theta(x) > 0$ for all $(\theta, x) \in A$. If there is (θ, x) with $\lambda^+(\theta, x) > 0$ ($\lambda^-(\theta, x) > 0$), then there exists an invariant graph ϕ in A with $\lambda(\mu_\phi) > 0$ ($\lambda(\mu_\phi) < 0$).*

Proof. For the case with $\lambda^+(\theta, x) > 0$, consider $M = A$, $T = f\restriction_A$, and $h\colon A \ni (\theta, x) \mapsto \log \partial_x f_\theta(x)$. After a similar computation as in (2.1.10), we see that the expression in (2.1.20) with $z = (\theta, x)$ is positively bounded away from zero for large enough n. Hence, due to Theorem 2.1.27, there must be a T-invariant and–without loss of generality–ergodic measure μ with $\int_A h\, d\mu > 0$. Note that μ straightforwardly extends to an invariant measure for f so that Theorem 2.1.21 and the corresponding remark in the previous section yield an invariant graph ϕ with $\mu = \mu_\phi$ and it is further not hard to see that ϕ is in A and $\int_A h\, d\mu_\phi = \lambda(\phi)$.

The case with $\lambda^-(\theta, x)$ works similarly with $T = f\restriction_A^{-1}$ and $h(\theta, x) = \log \partial_x f_\theta^{-1}(x)$. □

In the spirit of the discussion before Theorem 2.1.22, we see that points (θ, x) which converge to a *continuous* invariant graph ϕ have forward Lyapunov exponents $\lambda^+(\theta, x) =$

$\lambda(\phi)$. Hence, assuming the existence of continuous invariant graphs only, the following object should not exist.

Definition 2.1.29. A *sink-source orbit* is an orbit[3] whose backward and forward Lyapunov exponent is positive.

Theorem 2.1.30. ([Jäg09a, Theorem 2.4]) *Suppose $f\colon \Theta \times X \to \Theta \times X$ is a forced monotone C^1-interval map of the form* (2.1.1) *with $\partial_x f_\theta(x) > 0$ within a section $\Gamma = \Theta \times [\gamma^-, \gamma^+] \subseteq \Theta \times X$. Further, suppose there exists a sink-source-orbit entirely contained in Γ. Then there exist both an SNA and an SNR within Γ.*

Proof. Let $\mathcal{O}(\theta_0, x_0) = \{f^n(\theta_0, x_0)\colon n \in \mathbb{Z}\}$ be a sink-source orbit contained in Γ. Note that by continuity of f, $\overline{\mathcal{O}(\theta_0, x_0)} \subseteq \Gamma$ is a compact invariant set. Corollary 2.1.28 implies that there is a repeller and an attractor in $\overline{\mathcal{O}(\theta_0, x_0)}$. Denote the attractor by ϕ and observe that if ϕ was continuous, then $f^n(\theta_0, x_0) - \phi(\theta_0 + n\omega)$ would tend to 0 and $\lambda^+(\theta_0, x_0) = \lambda(\phi) < 0$ as discussed above and in contradiction to the hypothesis. The repeller can be dealt with similarly by sending n to $-\infty$. □

2.2. Fractal geometry: basic notions

From the introduction of [Fal03]: "My personal feeling is that the definition of a 'fractal' should be regarded in the same way as a biologist regards the definition of 'life'. There is no hard and fast definition, but just a list of properties characteristic of a living thing, such as the ability to reproduce or to move or to exist to some extent independently of the environment [...]. In the same way, it seems best to [...not] look for a precise definition which will almost certainly exclude some interesting cases."

In a similar spirit, we refrain from specifying the strangeness of SNA's in a precise way. Nevertheless, fractal geometry provides notions of dimension that allow to categorise sets with a *non-smooth* (in an admittedly vague sense) geometry. Two of the most common such dimensions are introduced in the next paragraph. For smooth manifolds, these dimensions coincide. For our SNA's, however, these notions happen to assume distinct values (see Theorem B)–supporting the naming *strange* attractor.

Similar to the different concepts of dimension of a set, we may study to which degree a given measure differs from measures supported on a "smooth" set. Respective concepts are presented in the last paragraph of this chapter. Apart from minor changes, the presentation of the following two paragraphs is as in the preliminary sections in [GJ13, FGJ14].

2.2.1. Hausdorff and box-counting dimension

In the following, let Y be a metric space. For $\varepsilon > 0$, we call a finite or countable collection $\{A_i\}$ of subsets of Y an *ε-cover* of A if $|A_i| \leq \varepsilon$ for each i and $A \subseteq \bigcup_i A_i$.

[3] We always consider full orbits when talking of sink-source orbits.

Definition 2.2.1. For $A \subseteq Y$, $s \geq 0$ and $\varepsilon > 0$, we define

$$\mathcal{H}^s_\varepsilon(A) = \inf\left\{ \sum_i |A_i|^s \;\middle|\; \{A_i\} \text{ is an } \varepsilon\text{-cover of } A \right\}$$

and call

$$\mathcal{H}^s(A) = \lim_{\varepsilon \to 0} \mathcal{H}^s_\varepsilon(A)$$

the *s-dimensional Hausdorff measure* of A. The *Hausdorff dimension* of A is defined by

$$D_H(A) = \sup\{s \geq 0 \mid \mathcal{H}^s(A) = \infty\}.$$

Lemma 2.2.2. *[Fal03, Section 2.1] Let $d \in \mathbb{N}$ and suppose $A \subseteq \mathbb{R}^d$ is a Borel set. Then*

$$\mathcal{H}^d(A) = 1/V_d \cdot \mathrm{Leb}_{\mathbb{R}^d}(A),$$

where V_d is the volume of the d-dimensional unit ball.

The proof of the next lemma is straightforward (cf. [GJ13, Lemma 2.7]).

Lemma 2.2.3. *Let $A \subseteq Y$ be a* lim sup *set, meaning that there exists a sequence $(A_i)_{i \in \mathbb{N}}$ of subsets of Y with*

$$A = \limsup_{i \to \infty} A_i = \bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} A_k.$$

If $\sum_{i=1}^{\infty} |A_i|^s < \infty$ for some $s > 0$, then $\mathcal{H}^s(A) = 0$ and $D_H(A) \leq s$.

Lemma 2.2.4 ([Fal03, page 32])**.** *Let Y be a metric space. The Hausdorff dimension is countably stable, that is, $D_H\left(\bigcup_i A_i\right) = \sup_i D_H(A_i)$ for any sequence of subsets $(A_i)_{i \in \mathbb{N}}$ with $A_i \subseteq Y$.*

Lemma 2.2.5 ([Fal03, Corollary 2.4])**.** *Let Y and Z be two metric spaces and assume that $g : A \subseteq Y \to Z$ is a bi-Lipschitz continuous map. Then $D_H(g(A)) = D_H(A)$.*

Definition 2.2.6. The *lower* and *upper box-counting dimension* of a totally bounded subset $A \subseteq Y$ are defined as

$$\underline{D}_B(A) = \liminf_{\varepsilon \to 0} \frac{\log N(A, \varepsilon)}{-\log \varepsilon},$$
$$\overline{D}_B(A) = \limsup_{\varepsilon \to 0} \frac{\log N(A, \varepsilon)}{-\log \varepsilon},$$

where $N(A, \varepsilon)$ is the smallest number of sets of diameter at most ε needed to cover A. If $\underline{D}_B(A) = \overline{D}_B(A)$, then we call their common value $D_B(A)$ the *box-counting dimension* (or *capacity*) of A.

Remark. In contrast to Lemma 2.2.4, we only have that the upper box-counting dimension is finitely stable [Fal03, Section 3.2, page 48]. Further, $D_B(A) = D_B\left(\overline{A}\right)$ [Fal03, Proposition 3.4].

Note that in [Fal03] Lemma 2.2.4, Lemma 2.2.5 and the last remark are formulated for subsets of $\mathbb{R}^d$ only. However, the proofs remain literally the same for subsets of general metric spaces.

Theorem 2.2.7 ([How96, Corollary 12] & [How95, Corollary 4])**.** *Suppose Y and Z are two metric spaces and consider the Cartesian product space $Y \times Z$ equipped with the maximum metric. Then for $A \subseteq Y$ and $B \subseteq Z$ totally bounded, we have*

$$D_H(A \times B) \leq D_H(A) + \overline{D}_B(B) \qquad \textit{and} \qquad D_H(A \times B) \geq D_H(A) + D_H(B).$$

Remark. In [How96, Corollary 12], the first of the above inequalities is actually formulated with $\overline{D}_B(B)$ replaced by the so-called *packing dimension* of B. We are not going to define the packing dimension (the interested reader may consult [How96, Section 2]). However, note that it is bounded from above by $\overline{D}_B(B)$ (see [Edg07, Proposition 6.8.8]).

2.2.2. Exact dimensional and rectifiable measures

We introduce the notions of pointwise and information dimension as well as exact dimensional measures. Further, we provide the definition and some properties of rectifiable measures where we mainly follow [AK00].

Again, let Y be a metric space. For $x \in Y$ and $\varepsilon > 0$ let $B_\varepsilon(x)$ be the open ball around x with radius $\varepsilon > 0$.

Definition 2.2.8. Suppose μ is a finite Borel measure in Y. For each point x in the support of μ, we define the *lower* and *upper pointwise dimension* $\underline{d}_\mu(x)$ and $\overline{d}_\mu(x)$, respectively, of μ at x by

$$\underline{d}_\mu(x) = \liminf_{\varepsilon \to 0} \frac{\log \mu(B_\varepsilon(x))}{\log \varepsilon},$$
$$\overline{d}_\mu(x) = \limsup_{\varepsilon \to 0} \frac{\log \mu(B_\varepsilon(x))}{\log \varepsilon}.$$

If $\underline{d}_\mu(x) = \overline{d}_\mu(x)$, then their common value $d_\mu(x)$ is called the *pointwise dimension* of μ at x. The *information dimension* of μ is defined by

$$\lim_{\varepsilon \to 0} \frac{\int \log \mu(B_\varepsilon(x))\, d\mu(x)}{\log \varepsilon},$$

provided the limit exists. Otherwise, we may define upper and lower information dimension via the limit superior and inferior, respectively.

Definition 2.2.9. We say that the measure μ is *exact dimensional* if the pointwise dimension exists and is constant almost surely, that is, we have

$$\underline{d}_\mu(x) = \overline{d}_\mu(x) = d_\mu$$

μ-almost surely.

Remark. Note that if μ is exact dimensional, then in the setting of separable metric spaces several other dimensions of μ coincide with the pointwise dimension [Zin02]. In particular, this is true for the information dimension [You82].

Definition 2.2.10. For $d \in \mathbb{N}$, we call a Borel set $A \subseteq Y$ *countably d-rectifiable* if there exists a sequence of Lipschitz continuous functions $(g_i)_{i\in\mathbb{N}}$ with $g_i : A_i \subseteq \mathbb{R}^d \to Y$ such that $\mathcal{H}^d(A \setminus \bigcup_i g_i(A_i)) = 0$. A finite Borel measure μ is called *d-rectifiable* if $\mu = \Theta\mathcal{H}^d\restriction_A$ for some countably d-rectifiable set A and some Borel measurable density $\Theta : A \to [0, \infty)$.

Observe that, by the Radon-Nikodym theorem, μ is d-rectifiable if and only if μ is absolutely continuous with respect to $\mathcal{H}^d\restriction_A$ where A is a countably d-rectifiable set.

Theorem 2.2.11 ([AK00, Theorem 5.4])**.** *For a d-rectifiable measure $\mu = \Theta\mathcal{H}^d\restriction_A$, we have*

$$\Theta(x) = \lim_{\varepsilon\to 0} \frac{\mu(B_\varepsilon(x))}{V_d \varepsilon^d}$$

$\mathcal{H}^d$-a.s. in A (where V_d is the volume of the d-dimensional unit ball).

The last theorem directly implies the next corollary.

Corollary 2.2.12. *A d-rectifiable measure μ is exact dimensional with $d_\mu = d$.*

3. Main results

After having provided the technical foundation of this thesis, we now summarise our main results. Despite the fact that the setting of saddle-node bifurcations–as introduced in the previous chapter–allows for a very general class of base maps, in order to prove the occurrence of a non-smooth bifurcation and further, to study the respective invariant graphs, we need a better control over the dynamics on the base Θ. For that reason, we consider the case where Θ is a torus (possibly higher dimensional) and the driving system is a minimal rotation which satisfies some additional slow recurrence property.

The first paragraph provides the result which ensures the occurrence of non-smooth saddle-node bifurcations for generic (in a sense specified below) families of skew product maps (Theorem A). The proof of Theorem A is given in Chapter 4. This result and its proof is the basis for all further investigation in this thesis.

The study of the topological and geometric properties of the considered SNA's relies heavily on an understanding of the combinatorial arguments developed in Chapter 4. The respective results are summarised in the second paragraph of this chapter. Their proofs are given in Chapter 5.

Finally, we treat skew product flow families by reducing them to appropriate skew product map families which fall into the regime of Theorem A. The result we obtain reads similarly to Theorem A and is given in the last paragraph.

3.1. Theorem A: genericity of strange attractors in discrete time

We consider the case of $\Theta = \mathbb{T}^d$ for some $d \in \mathbb{N}$ and assume the base map ω to be the rotation by a *rotation vector* ω–there is no risk of confusion since from now on ω always denotes an element in $\mathbb{T}^d$. The base map is thus given by $\mathbb{T}^d \ni \theta \mapsto \theta + \omega \in \mathbb{T}^d$. Altogether, we consider *quasiperiodically forced (qpf) monotone interval maps*

$$f\colon \mathbb{T}^d \times X \to \mathbb{T}^d \times X, \quad (\theta, x) \mapsto (\theta + \omega, f_\theta(x)), \tag{3.1.1}$$

where for each $\theta \in \mathbb{T}^d$ the fibre map $f_\theta : X \to X$ is assumed to be a *strictly* increasing interval map.

We denote by $\mathcal{F}_\omega(X)$ the class of qpf monotone C^2-interval-maps of the form (3.1.1) with fixed rotation vector $\omega \in \mathbb{T}^d$ in the base and fixed fibres X. Further by $\mathcal{P}_\omega(X)$, we denote the set of C^2-one-parameter families in $\mathcal{F}_\omega(X)$, that is,

$$\mathcal{P}_\omega(X) = \left\{ \left(f_\beta\right)_{\beta \in [0,1]} \;\middle|\; f_\beta \in \mathcal{F}_\omega(X) \text{ for all } \beta \in [0,1] \text{ and } (\beta, \theta, x) \mapsto f_{\beta,\theta}(x) \text{ is } C^2 \right\}.$$

Elements of $\mathcal{P}_\omega(X)$ are also denoted by $\hat{f} = (f_\beta)_{\beta\in[0,1]}$.

We endow $\mathcal{P}_\omega(X)$ with the extended metric

$$d(\hat{f},\hat{g}) = \sup_{\substack{(\theta,x)\in\mathbb{T}^d\times X\\ \beta\in[0,1]}} \sum_{\substack{s_1,s_2,s_3\in\{0,1,2\}\\ s_1+s_2+s_3\leq 2}} \left|\partial_\beta^{s_1}\partial_\theta^{s_2}\partial_x^{s_3} f_{\beta,\theta}(x) - \partial_\beta^{s_1}\partial_\theta^{s_2}\partial_x^{s_3} g_{\beta,\theta}(x)\right|.$$

With $\tilde{d} = d/(1+d)$, we may consider $\mathcal{P}_\omega(X)$ a metric space and refer to the respective topology as C^2-topology in all of the following.

We need the rotation vector to satisfy the following slow recurrence assumption. Note that as a result of this, the rotation on the base is minimal [KH97, Proposition 1.4.1].

Definition 3.1.1. We say $\omega \in \mathbb{T}^d$ is *Diophantine (of type $(\mathscr{C},\eta)$)* if there are $\mathscr{C} > 0$ and $\eta \in \mathbb{R}$ such that

$$\forall k \in \mathbb{Z}^d \setminus \{0\}\colon\ \sup_{p\in\mathbb{Z}} \left|\sum_{i=1}^{d} \omega_i k_i + p\right| \geq \mathscr{C}|k|^{-\eta}.$$

It is obvious that the hypothesis of Theorem 2.1.24 are fulfilled within a subset of $\mathcal{P}_\omega(X)$ with non-empty C^2-interior. It is thus natural to ask whether there is a subset $\mathcal{U}_\omega(X) \subseteq \mathcal{P}_\omega(X)$ with non-empty interior such that each $\hat{f} \in \mathcal{U}_\omega(X)$ undergoes a non-smooth bifurcation. The next statement gives an affirmative answer to this question and is the starting point for all further investigation in this thesis. It is thereby one of our three main results.

Theorem A. *Let X be an interval and suppose $\omega \in \mathbb{T}^d$ is Diophantine. There exists an open set $\mathcal{U}_\omega(X) \subseteq \mathcal{P}_\omega(X)$ such that each $\hat{f} \in \mathcal{U}_\omega(X)$ undergoes a non-smooth saddle-node bifurcation.*

Remark. In fact, the proof of Theorem A suggests that the considered C^2-openness is the most we can ask for in the sense that there should always be a C^1-close skew product family which undergoes a smooth bifurcation.

The set $\mathcal{U}_\omega(X)$ is specified in Chapter 4. Though its description may at first sight look rather technical, many of the assumptions a qpf map has to verify in order to lie in $\mathcal{U}_\omega(X)$ have a very natural origin and might even be considered necessary. Moreover, these assumptions are flexible enough to apply to simple explicit examples such as the family

$$f_\beta\colon \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R}, \qquad (\theta, x) \mapsto (\theta + \omega, \arctan(ax) - \beta \cdot (1 + \cos 2\pi\theta)),$$

with Diophantine ω and sufficiently large a (cf. Section 2.1.5, Figure 2.1 as well as the discussion in Section 4.1). Note that the important features of this example are not so much the particular choice of the fibre maps but rather general properties like the concavity[1] of the fibre maps, the existence of regions with strong vertical expansion as

[1] Of course, (strict) concavity is only guaranteed for $x > 0$.

well as of regions with strong vertical contraction (therefore the assumption of large a), a decreasing dependence on β, and a "bump-like" dependence on θ, that is, the image of any line $\mathbb{T}^d \times \{x\}$ under f_β has a unique, non-degenerate global minimum for $\beta > 0$.

In this sense, Theorem A allows for a much ampler application (not just abstractly, but practically) than prior results which are applicable only in special families of qpf monotone interval maps [Bje07a, Jäg09a]: [Bje07a] considers the particular case of the Harper map, which is given by the projective action of a certain class of $\mathrm{SL}(2,\mathbb{R})$-cocycles, so-called *Schrödinger cocycles*; [Jäg09a] yields a result for additive forcing by imposing a non-differentiability assumption on the forcing term and thereby excludes the application to smooth examples.

3.2. Geometry of strange attractors

Having thus a good understanding of the occurrence of non-smooth bifurcations, it is natural to study the geometry of the invariant graphs at the bifurcation parameter β_c. From the mathematical viewpoint, much of the relevant information about the geometric and dynamical features of an attractor is encoded in different notions of dimension. Accordingly, the question of computing dimensions of SNA's has been raised already at an early stage. Based on numerical evidence and heuristic arguments, it has been conjectured in [DGO89] that the box-counting dimension of SNA's appearing in different types of qpf systems with one-dimensional base $\mathbb{T}^1$ and one-dimensional fibres equals two, whereas the information dimension equals one. For the simple pinched skew products introduced in [GOPY84], these findings were confirmed analytically in [Jäg07, GJ13].

Apart from the dimensions at the bifurcation point, we obtain a simple description of the minimal set in the section $\Gamma = \mathbb{T}^d \times [\gamma^-, \gamma^+]$ (recall that we are dealing with local bifurcations) as the *maximal invariant set (in* Γ*)*. For $\beta \in [0, 1]$, this is given by

$$\Lambda_\beta = \bigcap_{n\in\mathbb{Z}} f_\beta^n(\Gamma).$$

Note that Λ_β is non-empty for $\beta \leq \beta_c$. By Lemma 2.1.10,

$$\phi^-_{\Lambda_\beta}(\theta) = \inf \Lambda_\beta(\theta) \quad \text{and} \quad \phi^+_{\Lambda_\beta}(\theta) = \sup \Lambda_\beta(\theta)$$

are lower and upper semi-continuous invariant graphs, respectively. They are thus representatives of the invariant graphs that appear along the saddle-node bifurcation of $(f_\beta)_{\beta\in[0,1]}$.

Theorem B. *Let $\omega \in \mathbb{T}^d$ be Diophantine. Then there exists $\mathcal{V}_\omega(X) \subseteq \mathcal{U}_\omega(X)$ with non-empty C^2-interior such that for all $\hat{f} \in \mathcal{V}_\omega(X)$ the SNA $\phi^+_{\Lambda_{\beta_c}}$ appearing at the critical bifurcation parameter β_c satisfies the following.*

(i) $D_B\left(\Phi^+_{\Lambda_{\beta_c}}\right) = d + 1$ *and* $D_H\left(\Phi^+_{\Lambda_{\beta_c}}\right) = d$.

(ii) The measure $\mu_{\phi^+_{\Lambda_{\beta_c}}}$ is d-rectifiable with pointwise dimension and information dimension equal to d.

(iii) The set $\Lambda_{\beta_c} = \left[\phi^-_{\Lambda_{\beta_c}}, \phi^+_{\Lambda_{\beta_c}}\right]$ is minimal. We have $\Lambda_{\beta_c} = \overline{\Phi^-_{\Lambda_{\beta_c}}} = \overline{\Phi^+_{\Lambda_{\beta_c}}}$.

(iv) $\phi^+_{\Lambda_{\beta_c}}$ is the only semi-continuous representative in its equivalence class.

Analogous results hold for the repeller $\phi^-_{\Lambda_{\beta_c}}$.

On an heuristic level, some inspiration for the computation of $D_H(\Phi^+_{\Lambda_{\beta_c}})$ and the proof of (ii) is drawn from [GJ13]. Technically, however, the task is considerably more demanding and our approach builds on the detailed multiscale analysis established in the proof of Theorem A.

Item (iii) in Theorem B has already been considered by M. Herman [Her83]. We want to mention that it has been proved previously by Bjerklöv for invariant graphs appearing in the Harper map [Bje07a], which can be considered a special case of our setting. Our proof is inspired by that of Bjerklöv, but puts a stronger focus on the global approximation of the SNA by iterates of continuous curves. This allows to avoid some technical complications. The strategy of our proof is outlined at the beginning of Section 5.2.

We also note that the result on the box-counting dimension is a direct consequence of (iii). Since the box-counting dimension is stable under taking closures, we have $D_B\left(\Phi^+_{\Lambda_{\beta_c}}\right) = D_B(\Lambda_{\beta_c})$. Since the boundary graphs of Λ_{β_c} are almost surely distinct, this set has positive $d + 1$-dimensional Lebesgue measure and therefore box-counting dimension $d + 1$.

3.3. Genericity of strange attractors in continuous time

Let $\Theta = \mathbb{T}^D$ for some integer $D \geq 2$. We assume the driving system to be a rotation with rotation vector $\rho \in \mathbb{R}^D$. Altogether, we consider so-called *quasiperiodically forced (qpf) (local) flows* corresponding to some non-autonomous vector field $F\colon \mathbb{T}^D \times X \to \mathbb{R}$

$$\Xi\colon U \subseteq \mathbb{R} \times \mathbb{T}^D \times X \to \mathbb{T}^D \times X, \quad (t, \theta, x) \mapsto (t \cdot \rho + \theta, \xi(t, \theta, x)), \tag{3.3.1}$$

where ξ is the maximal solution of

$$\partial_t \xi(t, \theta, x) = F\left(t \cdot \rho + \theta, \xi(t, \theta, x)\right)$$

with $\xi(0, \theta, x) = x$ for each $(\theta, x) \in \mathbb{T}^D \times X$, and U is the domain of ξ. For questions about existence and uniqueness of $\xi(\cdot, \theta, x)$, the reader is referred to the preliminary observations in Section 2.1. Again, we assume ρ to verify a slow recurrence assumption.

Definition 3.3.1. We say $\rho \in \mathbb{R}^D$ is *Diophantine (of type $(\mathscr{C}, \eta)$)* if there are $\mathscr{C} > 0$ and $\eta \in \mathbb{R}$ such that

$$\forall k \in \mathbb{Z}^D \setminus \{0\}\colon \left|\sum_{i=1}^{D} \rho_i k_i\right| \geq \mathscr{C}|k|^{-\eta}.$$

We denote the set of non-autonomous C^2-vector-fields on $\mathbb{T}^D \times X$ by $\mathscr{F}(X)$ (keeping the dimension D implicit). The set of C^2-one-parameter families in $\mathscr{F}(X)$ is denoted by

$$\mathscr{P}(X) = \left\{ \left(F_\beta\right)_{\beta\in[0,1]} \middle| F_\beta \in \mathscr{F}(X) \text{ for all } \beta \in [0,1] \text{ and } (\beta, \theta, x) \mapsto F_\beta(\theta, x) \text{ is } C^2 \right\}.$$

We may denote elements of $\mathscr{P}(X)$ also by $\hat{F} = (F_\beta)_{\beta\in[0,1]}$. Similarly to the discrete time case, we consider $\mathscr{P}(X)$ endowed with the extended metric

$$d(\hat{F}, \hat{G}) = \sup_{\substack{(\theta,x)\in\mathbb{T}^D\times X \\ \beta\in[0,1]}} \sum_{\substack{s_1,s_2,s_3\in\{0,1,2\} \\ s_1+s_2+s_3\leq 2}} \left|\partial_\beta^{s_1}\partial_\theta^{s_2}\partial_x^{s_3} F_\beta(\theta, x) - \partial_\beta^{s_1}\partial_\theta^{s_2}\partial_x^{s_3} G_\beta(\theta, x)\right|$$

and refer to the respective topology as C^2-topology.

Theorem C. *Suppose $\rho \in \mathbb{R}^D$ is Diophantine. Then there is a set $\mathscr{U}_\rho(X) \subseteq \mathscr{P}(X)$ with non-empty interior in the C^2-topology such that each family of flows $(\Xi_\beta)_{\beta\in[0,1]}$ driven by $(t, \theta) \mapsto t \cdot \rho + \theta$ and generated by some $(F_\beta)_{\beta\in[0,1]} \in \mathscr{U}_\rho(X)$ undergoes a non-smooth saddle-node bifurcation.*

So far, all rigorous results on the existence of SNA's of skew product *flows* are provided by rather particular examples of projective actions of linear cocycles [Vin75, Joh82, Bje07b], see also [Mil68, Kol87, Lip00, JTNO07]. While [Bje07b] treats the case beyond the saddle-node bifurcation scenario, [Joh82] (which discusses the occurrence of an SNA/SNR-pair in an example provided in [Vin75]) makes extensive use of the machinery developed for the study of linear differential equations (cf. [SS78], for example).

The essence of the proof of Theorem C is to show that–given a Diophantine rotation vector–the family of qpf skew product flows corresponding to the non-autonomous vector fields

$$F_\beta(\theta, x) = -bx^2 + b - \beta b \cdot g(\theta),$$

where g is a bump function with a unique, non-degenerate maximum and b is assumed to be large, admits a Poincaré section such that the corresponding first return maps lie in the set $\mathcal{U}_\omega(X)$ (cf. Section 6.2) or even in $\mathcal{V}_\omega(X)$ (with some suitable $\omega = \omega(\rho)$). As before, it is not so much the particular choice of the vector fields F_β but the assumption of general features–like the concavity of the functions $F_\beta(\theta,\cdot)$ and the decreasing dependence on β–which allow for an application of Theorem A.

Hence, on a technical level, there is a big difference between our result and the earlier ones. Further, by the application of Theorem A, our approach focuses on the geometry of the mechanism by which SNA's are created. This geometric insight shows that even if in general, analytical results on the occurrence of non-smooth bifurcations might still be subject to rather technical considerations, the proof in Section 6.2 should basically be extendable to situations with non-autonomous vector fields similar to the one above (for example, if x^2 is replaced by another function $h\colon \mathbb{R} \to \mathbb{R}$ with a positive lower bound on the second derivatives).

The genericity of non-smooth bifurcations in the continuous time case (that is, the non-empty interior of $\mathscr{U}_\rho(X)$) comes as a by-product of the application of Theorem A.

Further, almost no extra work is needed in order to carry over the geometric results of the previous section to the continuous time case. More precisely, Theorem B turns out to remain true with D instead of d and the maximal invariant set $\tilde{\Lambda}_{\beta_c} = \bigcap_{\tau\in\mathbb{R}} \Xi_{\beta_c}(\{\tau\}\times\Gamma)$ if we replace the boundary graphs $\phi^{\pm}_{\Lambda_{\beta_c}}$ and their associated measures $\mu_{\phi^{\pm}_{\Lambda_{\beta_c}}}$ by the boundary graphs $\psi^{\pm}_{\tilde{\Lambda}_{\beta_c}}$ of $\tilde{\Lambda}_{\beta_c}$ and their corresponding measures $\mu_{\psi^{\pm}_{\tilde{\Lambda}_{\beta_c}}}$, respectively.

4. Existence of strange attractors in forced monotone interval maps

It not just reflects the chronological order of the results of this thesis, but also follows the logical thread running through this work to first present that statement which asserts the (C^2-)genericity of non-smooth saddle-node bifurcations. This is what the current chapter is about.

In this sense, we set the framework for all the further discussion here: in the subsequent chapters, SNA's will always be SNA's created in the way described in this section.

Therefore, a rather comprehensive description of the set $\mathcal{U}_\omega(X)$ of Theorem A is desirable. To a large extent, this description is given in the first paragraph. In the second paragraph, we prove Theorem A.

4.1. Theorem A revisited

In this section, we specify the set $\mathcal{U}_\omega(X)$ of Theorem A to a large degree. Note that the actual definition is provided at the end of this chapter (see Definition 4.2.16). It is essentially a collection of assumptions on the derivatives of the fibre maps $f_{\beta,\theta}$ up to second order. In order to both make the reader familiar with these assumptions and to demonstrate how they apply to some standard skew product families, we explicitly show that these assumptions are met by

$$f_\beta \colon \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R}, \qquad (\theta, x) \mapsto (\theta + \omega, \arctan(ax) - \beta \cdot (1 + \cos 2\pi\theta)), \qquad (*)$$

for Diophantine $\omega \in \mathbb{T}^1$ and large enough a where $\beta \in [0, 1]$ (cf. Section 2.1.5, Figure 2.1). Observe that $(*)$ lies in $\mathcal{P}_\omega(\mathbb{R})$.

From the discussion below, it becomes clear how an analogous example acting on $\mathbb{T}^d \times X$ with arbitrary $d \in \mathbb{N}$ and an arbitrary interval $X \subseteq \mathbb{R}$ can be constructed. This shows that the set $\mathcal{U}_\omega(X)$ is indeed non-empty. However, in the interest of clarity, we restrict to the particular family $(*)$ here. Examples with arbitrary base dimension can further be found in Chapter 6.

For now, we provide the description of a set $\widehat{\mathcal{U}}_\omega(X) \subseteq \mathcal{P}_\omega(X)$, such that each $\hat{f} \in \widehat{\mathcal{U}}_\omega(X)$ allows for a parameter at which an SNA/SNR pair occurs. In other words, not every $\hat{f} \in \widehat{\mathcal{U}}_\omega(X)$ necessarily undergoes a saddle-node bifurcation. As the assumptions that define $\widehat{\mathcal{U}}_\omega(X)$ are compatible with (2.1.11)–(2.1.15), we can basically think of $\mathcal{U}_\omega(X)$ as the collection of those $\hat{f} \in \widehat{\mathcal{U}}_\omega(X)$ which undergo a saddle-node bifurcation, which then is necessarily non-smooth.

Due to Theorem 2.1.30, it suffices to provide a sink-source orbit in order to show the existence of an SNA/SNR pair. To find such an orbit, we search for trajectories that spend most of the positive times in *regions of (vertical) expansion* $\mathbb{T}^d \times E$ with $E = [e^-, e^+]$ and most of the negative times in *(vertically) contracting regions* $\mathbb{T}^d \times C$ with $C = [c^-, c^+]$ where we assume the expanding region to lie below the contracting region,[1] that is, $e^+ < c^-$. Despite the fact that such trajectories spend long times in either of these regions, we obviously need a connection between $\mathbb{T}^d \times E$ and $\mathbb{T}^d \times C$. This is why we only consider parameters not smaller than $\beta_-(0) = \min\{\beta \in [0,1]\colon \exists \theta \in \mathbb{T}^d$ such that $f_{\beta,\theta}(c^-) \leq e^+\}$ in the following.[2]

On the other hand, note that family $(*)$ has an attracting invariant graph below $\mathbb{T}^d \times \mathbb{R}_{\geq 0}$. If β is too large, all orbits eventually tend to this graph. This corresponds to the case $\beta > \beta_c$ of a local saddle-node bifurcation (occurring in the section with non-negative x-values). We set $\beta_+(0) = \max\{\beta \in [0,1]\colon f_{\beta,\theta}(c^+) \geq e^-$ for all $\theta \in \mathbb{T}^d\}$. Note that in general the well-definition of $\beta_-(0)$ and $\beta_+(0)$ follows from assumption $(\mathcal{A}7)$ below.

Putting $\mathscr{B}(0) = [\beta_-(0), \beta_+(0)]$, the following assumptions are thus supposed to hold for all $\beta \in \mathscr{B}(0)$ (if applicable).

In order to ensure that $\mathbb{T}^d \times C$ and $\mathbb{T}^d \times E$ are indeed regions of vertical contraction and expansion, respectively, we assume

($\mathcal{A}1$) $0 < \partial_x f_{\beta,\theta}(x) < \alpha_c$ for $(\theta, x) \in \mathbb{T}^d \times C$,

($\mathcal{A}2$) $\partial_x f_{\beta,\theta}(x) > \alpha_e$ for $(\theta, x) \in (\mathbb{T}^d \setminus \mathcal{I}_0) \times E \bigcap f_\beta^{-1}(\mathbb{T}^d \times E)$,

where $0 < \alpha_c < 1 < \alpha_e$ and $\mathcal{I}_0 \subseteq \mathbb{T}^d$ is the so-called *(0-th) critical region* which is specified below. We restrict our analysis to the section $\mathbb{T}^d \times [e^-, c^+]$ in which

($\mathcal{A}3$) $\alpha_l < \partial_x f_{\beta,\theta}(x) < \alpha_u$ for all $(\theta, x) \in \mathbb{T}^d \times [e^-, c^+] \cap f_\beta^{-1}(\mathbb{T}^d \times [e^-, c^+])$,

where $0 < \alpha_l < \alpha_u$. Observe that c^+ and e^- play the role of γ^+ and γ^- in Theorem 2.1.24, respectively,

($\mathcal{A}4$) $f_{\beta,\theta}(c^+) \leq c^+$ and $f_{\beta,\theta}(e^-) \leq e^-$.

As already mentioned, we need a connection between the contracting and expanding region. This connection is provided by the critical region $\mathcal{I}_0$. More precisely,

($\mathcal{A}5$) $f_{\beta,\theta}(x) \in C$ for all $x \in [e^+, c^+]$ and $\theta \notin \mathcal{I}_0$

and we assume there is a subset $\mathcal{J}_{0,\beta} \subseteq \mathcal{I}_0$ containing (at least) all $\theta \in \mathcal{I}_0$ for which $f_{\beta,\theta}(c^-) \leq e^+$ with

($\mathcal{A}6$) $\mathcal{J}_{0,\beta}$ is closed and convex and $\mathcal{J}_{0,\beta} \subseteq \mathcal{J}_{0,\beta'}$ for $\beta \leq \beta'$.

[1]This implicitly introduces the convention that the attractor does not lie below the repeller at any θ. Of course, a symmetric version with C below E, that is, with the attractor below the repeller holds true as well.

[2]Note that we assume a monotonously decreasing dependence on the family parameter β (cf. ($\mathcal{A}8$)).

Note that due the monotonicity and by means of ($\mathcal{A}4$), assumption ($\mathcal{A}5$) is equivalent to

($\mathcal{A}5'$) $f_{\beta,\theta}^{-1}(x) \in E$ for all $x \in [e^-, c^-]$ and $\theta \notin \mathcal{I}_0 + \omega$.

Theorem 2.1.24 shows that the mechanism by which the SNA/SNR pair is created in a saddle-node bifurcation is essentially the following: First, the existence of a continuous attractor and repeller is guaranteed for small β. Then, by increasing β, we move these two initial invariant graphs closer and closer to each other on a small set until they finally touch on a $\mathrm{Leb}_{\mathbb{T}^d}$-null set which results in the discontinuity. The existence of the initial invariant graphs is ensured by ($\mathcal{A}4$) and the first part of

($\mathcal{A}7$) $f_{0,\theta}(c^-) \geq c^-$ for all $\theta \in \mathbb{T}^d$ and $f_{\beta_+(0),\theta}(c^+) \leq e^-$ for some $\theta \in \mathbb{T}^d$,

while the second part–which should be understood as "there is $\beta_+(0) \leq 1$ such that orbits can cross the section $\mathbb{T}^d \times [e^-, c^+]$ while ($\mathcal{A}1$)–($\mathcal{A}6$) remain true for all $\beta \leq \beta_+$"–guarantees the well-definition of $\beta_-(0)$ and $\beta_+(0)$.

Before formulating further assumptions, let us define the introduced quantities for ($*$) and see how ($\mathcal{A}1$)–($\mathcal{A}7$) are verified in this particular case. Set $e^- = 0$, $e^+ = r/a$ for some $r > 1$, $c^- = 1/r$ and $c^+ = \pi/2$. Note that $e^- < e^+ < c^- < c^+$ for large enough a. As

$$\partial_x f_{\beta,\theta}(x) = \frac{a}{1 + (ax)^2},$$

we get ($\mathcal{A}1$)–($\mathcal{A}3$) with $\alpha_e, \alpha_c^{-1} = a^{2/p}$ and $\alpha_u, \alpha_l^{-1} = a^p$ for any fixed $p > 2$ if a is large enough. Further, ($\mathcal{A}4$) is evident and ($\mathcal{A}7$) holds trivially with $\beta_+(0) \leq \pi/4$ under the assumption of large enough a. Finally, a natural choice for the critical region is given by

$$\mathcal{I}_0 = \left\{\theta \in \mathbb{T}^1 : f_{\beta,\theta}(e^+) \leq c^-\right\} = \left\{\theta \in \mathbb{T}^1 : \cos 2\pi\theta \geq (\arctan r - 1/r)/\beta - 1\right\}. \quad (4.1.1)$$

With $\mathcal{J}_{0,\beta} = \mathcal{I}_0$, we see that ($\mathcal{A}5$) and ($\mathcal{A}6$) are verified by definition.

As in Theorem 2.1.24, we assume monotone dependence on β.

($\mathcal{A}8$) $f_{(\cdot)}(\theta, x)$ is non-increasing for fixed $(\theta, x) \in \mathbb{T}^d \times [e^-, c^+]$.

Observe that ($\mathcal{A}8$) is trivially fulfilled by ($*$).

The critical region $\mathcal{I}_0$ allows jumps from the contracting to the expanding region. On the other hand, we also want the sink-source orbit to spend long times in the respective regions without jumping out too often, that is, we don't want $\mathcal{J}_{0,\beta}$ to be too big. In (4.1.1), we see that by choosing large r, we can make $\mathcal{J}_{0,\beta}$ arbitrarily small for large enough a. This results from the fact that the second derivative $\partial_\theta^2 f_{\beta,\theta}(x) = 4\pi^2\beta \cdot \cos 2\pi\theta$ is bounded away from 0 on $\mathcal{J}_{0,\beta}$. In general, we thus assume there exists $s > 0$ such that

($\mathcal{A}9$) $\partial_\vartheta^2 f_{\beta,\theta}(x) > s$ for each $\vartheta \in \mathbb{S}^{d-1}$, $x \in C$ and all $\theta \in \mathcal{J}_{0,\beta}$.

To motivate further assumptions, we need to provide a rough sketch of how to prove the existence of a sink-source orbit. Assuming that $\mathcal{I}_0$ is small, there is a positive number

M_0 such that the first M_0 forward and backward iterates of $\mathcal{I}_0 + \omega$ under the base transformation (that is, under the rigid rotation with rotation vector ω) do not intersect, so that in particular

$$\mathcal{I}_0 + \omega \cap \bigcup_{k=\pm 1,\ldots,\pm M_0} (\mathcal{I}_0 + (k+1)\omega) = \emptyset.$$

If this is true, $f_\beta^l(\theta, x)$ never leaves the contracting region for $l = 0, \ldots, M_0 - 1$ if $\theta \in \mathcal{I}_0 - (M_0 - 1)\omega$, $x \in C$, while $f_\beta^{-l}(\theta, x)$ never leaves the expanding region for $l = 0, \ldots, M_0$ if $\theta \in \mathcal{I}_0 + (M_0+1)\omega$, $x \in E$, assuming ($\mathcal{A}$5) and ($\mathcal{A}$5′). However, $f_\beta^{M_0-1}(\theta, x)$ might jump into the expanding region in the next iteration of f_β or, even fall into the set $f_\beta^{-M_0}([\mathcal{I}_0 + (M_0+1)\omega] \times E)$. In the latter case, $f_\beta^{M_0-1}(\theta, x)$ is a first candidate for a sink-source orbit as it stays in the expanding region for $M_0 + 1$ times while its backward iterates stay in the contracting region for $M_0 - 1$ times.

The projection of the set of all such sink-source orbit candidates to the base $\mathbb{T}^d$ is denoted by $\mathcal{I}_1$. Similarly as in the case of $\mathcal{I}_0$, we need that $\mathcal{I}_1$ is small enough to guarantee long return times. However, we actually need it to visit itself with an even smaller frequency than $\mathcal{I}_0$ does. To that end, we need that the second derivatives of the maps $\mathcal{J}_{0,\beta} \ni \theta \mapsto f^{M_0}_{\beta,\theta-M_0\omega}(x) - f^{-M_0}_{\beta,\theta+M_0\omega}(y)$ (for fixed $x \in C$ and $y \in E$) are positive–similar to the second derivatives of $f_{\beta,\theta}$ in ($\mathcal{A}$9). It will turn out that this amounts to controlling all the other derivatives of f_β and its inverse[3] (see the proof of Lemma 4.2.13). Let $S > 0$ be such that

($\mathcal{A}$10) $\left|\partial_\vartheta f_{\beta,\theta}(x)\right| < S$ for all $(\theta, x) \in \mathbb{T}^d \times [e^-, c^+] \bigcap f_\beta^{-1}(\mathbb{T}^d \times [e^-, c^+])$ and $\vartheta \in \mathbb{S}^{d-1}$,

($\mathcal{A}$11) $\left|\partial^2_\vartheta f_{\beta,\theta}(x)\right| < S^2$ for all $(\theta, x) \in \mathbb{T}^d \times [e^-, c^+] \bigcap f_\beta^{-1}(\mathbb{T}^d \times [e^-, c^+])$ and $\vartheta \in \mathbb{S}^{d-1}$,

($\mathcal{A}$12) $\left|\partial_\vartheta \partial_x f_{\beta,\theta}(x)\right| < \begin{cases} S\,\alpha_c & \text{for } (\theta, x) \in \mathbb{T}^d \times C \\ S\,\alpha_u^2 & \text{for } (\theta, x) \in \mathbb{T}^d \times [e^-, c^+] \bigcap f_\beta^{-1}(\mathbb{T}^d \times [e^-, c^+]) \end{cases}$ for $\vartheta \in \mathbb{S}^{d-1}$.

Further, suppose

($\mathcal{A}$13) $\left|\partial^2_x f_{\beta,\theta}(x)\right| < \begin{cases} \alpha_c & \text{for } (\theta, x) \in \mathbb{T}^d \times C \\ \alpha_u^2 & \text{for } (\theta, x) \in \mathbb{T}^d \times [e^-, c^+] \bigcap f_\beta^{-1}(\mathbb{T}^d \times [e^-, c^+]). \end{cases}$

For the derivatives of the inverse, we get similar estimates by means of the inverse function theorem. However, we additionally need

($\mathcal{A}$14) $\left|\partial^2_x f^{-1}_{\beta,\theta}(x)\right| < \alpha_e^{-1}$ for each $\theta \notin \mathcal{I}_0 + \omega$ and $x \in E$,

($\mathcal{A}$15) $\left|\partial_\vartheta \partial_x f^{-1}_{\beta,\theta}(x)\right| < S\,\alpha_e^{-1}$ for each $\theta \notin \mathcal{I}_0 + \omega$, $x \in E$ and $\vartheta \in \mathbb{S}^{d-1}$.

[3] Notice that due to the inverse function theorem and ($\mathcal{A}$3), there is an open neighbourhood U of $\mathbb{T}^d \times [e^-, c^+] \cap f_\beta^{-1}(\mathbb{T}^d \times [e^-, c^+])$ so that $(f_\beta \restriction_U)^{-1}$ is C^2.

Coming back to $(*)$, we get ($\mathcal{A}$10) and ($\mathcal{A}$11) with $S = \max_{\beta,\theta,x} \partial_\theta f_{\beta,\theta}(x) = 2\pi$. ($\mathcal{A}$12) is trivial, as mixed derivatives vanish. With

$$\partial_x^2 f_{\beta,\theta}(x) = \frac{-2a^3 x}{(1+(ax)^2)^2},$$

we get $|\partial_x^2 f_{\beta,\theta}(x)| < a^{-2/p}$ for big enough a and $x \in C$. Further, basic calculus yields $|\partial_x^2 f_{\beta,\theta}(x)| \leq \left|\partial_x^2 f_{\beta,\theta}\left(\sqrt{1/(3a^2)}\right)\right| = O(a^2)$ as $a \to \infty$. This shows ($\mathcal{A}$13) for big enough a. If $x \in E$ and $\theta \notin \mathcal{I}_0 + \omega$, we moreover have

$$\begin{aligned}
\partial_x^2 f_{\beta,\theta}^{-1}(x) &= 2/a \cdot \frac{\sin(x + \beta\cdot(\cos 2\pi(\theta-\omega)+1))}{\cos^3(x+\beta\cdot(\cos 2\pi(\theta-\omega)+1))},\\
\partial_\theta\partial_x f_{\beta,\theta}^{-1}(x) &= -4\pi\beta/a \cdot \frac{\sin 2\pi(\theta-\omega)\cdot\sin(x+\beta\cdot(\cos 2\pi(\theta-\omega)+1))}{\cos^3(x+\beta\cdot(\cos 2\pi(\theta-\omega)+1))}.
\end{aligned}$$

For $\theta \notin \mathcal{I}_0 + \omega$, (4.1.1) yields $\beta\cdot(\cos 2\pi(\theta-\omega)+1) < \arctan r - c^-$ which proves ($\mathcal{A}$14) and ($\mathcal{A}$15) for large enough a, since $0 \leq x \leq e^+ < c^-$.

We are now in a position to state the main theorem of this chapter.

Theorem 4.1.1. *Suppose $\omega \in \mathbb{T}^d$ is Diophantine of type $(\mathscr{C},\eta)$, $X \subseteq \mathbb{R}$ is some non-degenerate interval and $(f_\beta)_{\beta\in[0,1]} \in \mathcal{P}_\omega(X)$ satisfies ($\mathcal{A}$1)–($\mathcal{A}$15). Let there be $p \geq \sqrt{2}$ and $\alpha > 1$ with*

$$\alpha_c^{-1} = \alpha_e = \alpha^{2/p}, \qquad \alpha_l^{-1} = \alpha_u = \alpha^p.$$

Further, suppose $|\mathcal{I}_0| < \varepsilon_0$ and $\alpha > \alpha_0$ for some strictly positive constants $\varepsilon_0 = \varepsilon_0(p,\mathscr{C},\eta)$ and $\alpha_0 = \alpha_0(s,S,p,|C|,|E|,\mathscr{C},\eta)$. Then there is $\beta_c \in [0,1]$ such that f_{β_c} has a sink-source orbit, and hence an SNA and an SNR in $\mathbb{T}^d \times [e^-,c^+]$.

Remark. (a) α_0 can be chosen to be non-decreasing in $|C|$ and $|E|$.

(b) For later use, we provide another formulation of the above theorem at the end of this chapter (Theorem 4.2.15), which also applies in cases when S and s depend on α.

Setting $\alpha = a$, the previous discussion shows that $(*)$ satisfies the hypothesis of Theorem 4.1.1 if a is large enough. An important step towards the understanding of the creation of SNA's was the verification of a non-smooth saddle-node bifurcation for the Harper map

$$(\theta,x) \mapsto \left(\theta+\omega, \arctan\left(\frac{-1}{\tan(x) - E + \lambda V(\theta)}\right)\right),$$

which is closely related to the discrete quasi-periodic Schrödinger equation. In [Bje07a] it is shown that if the *potential* V is C^2 and if it assumes its unique global maximum at a point with non-vanishing second derivative, then we observe a non-smooth saddle node-bifurcation upon a decrease of E if λ is large enough.

The geometric idea of our proof is inspired by the proof in [Bje07a] as can be readily seen from Figure 4.1 and Figure 4.2 in Section 4.2.2. It is thus not surprising that we can recover Bjerklov's result with the same regularity assumptions. However, as we do not restrict to fibre maps of a particular shape, more work is needed in order to get control over the sink-source orbit. This extra work is certainly best visible in the proof of Lemma 4.2.13.

Despite the fact that ($\mathcal{A}1$)–($\mathcal{A}15$) seem rather technical, they just capture the main qualitative properties of some standard examples which posses an SNA and turn out to be flexible enough to treat different skew product families at the same time. We have seen that ($*$) verifies the assumptions of Theorem 4.1.1. As a generalisation of the arctan-family ($*$), for each $q > 1$ we can apply Theorem 4.1.1 to

$$(\theta, x) \mapsto \left(\theta + \omega, h_q(\alpha x) - \beta h_q(\infty) \cdot (1 + \cos 2\pi\theta)\right),$$

where $h_q(x) = \operatorname{sgn}(x) \cdot \tilde{h}_q(|x|)$ with $\tilde{h}_q(x) = \int_0^x (1 + \zeta^q)^{-1}\, d\zeta$, which can be seen similarly as for ($*$). Analogously, we obtain the existence of an SNA for the family

$$(\theta, x) \mapsto h_q(\alpha x) - 2\beta - (1 + \sin 2\pi\theta)/2,$$

which has been considered numerically for $q = 2$ in [AJ12].

Remark. The assumption that $\alpha_c^{-1} = \alpha_e = \alpha^{2/p}$ and $\alpha_l^{-1} = \alpha_u = \alpha^p$ is for technical reasons. It basically originates from the fact that we define $\mathcal{I}_1$ in a symmetric way, that is, we consider the intersection of the M_0^--th iterate of $\mathcal{I}_0 - (M_0^- - 1)\omega \times C$ and the M_0^+-th inverse iterate of $\mathcal{I}_0 + (M_0^+ + 1)\omega \times E$ with $M_0^+ = M_0^-$ (see Definition 4.2.1 and Definition 4.2.2 for the precise formulation). We believe that by allowing different relations between M_0^+ and M_0^-, we can also allow different scaling behaviour in order to apply a similar statement like Theorem 4.1.1 to $(\theta, x) \mapsto (\theta + \omega, \tanh(\alpha x) - \beta[1 + \cos(2\pi\theta)])$, for example, where the ratio of α_l^{-1}/α_u grows exponentially with α.

4.2. Proof of Theorem A

In this section, we prove Theorem 4.1.1 by showing that there is a point (θ, x) whose positive iterates mostly stay in the expanding region, while its negative iterates mostly stay in the contracting region. This can be achieved if the frequency of the jumps from one region to the other is small enough, which is the idea behind the inductive assumptions $(\mathcal{F}1)_n$ and $(\mathcal{F}2)_n$ (Section 4.2.1). These are basically hypothesis on the size of the inductively defined *critical regions* $\mathcal{I}_n$. By a geometric argument, we will get upper bounds for these quantities in Section 4.2.2. In Section 4.2.3, we eventually show that these upper bounds decrease fast enough to guarantee the existence of an SNA.

4.2.1. Combinatorial considerations

We make use of ($\mathcal{A}1$)–($\mathcal{A}5$) to estimate the vertical growth rate of orbits which converge to a sink-source orbit. In order to achieve this, we need to assume some additional inductive assumptions. The verification of these additional assumptions is the goal of the subsequent sections. As a matter of fact, the statements of this paragraph are basically provided in [Bje07a, Jäg09b], already. For the convenience of the reader and as there are a number of subtle technical differences, we nevertheless include their proofs here.

In the following, let $(M_n)_{n\in\mathbb{N}_0}, (K_n)_{n\in\mathbb{N}_0} \in \mathbb{N}^{\mathbb{N}_0}$ be strictly increasing sequences with $M_0 \geq 2$ and $M_n \leq 2K_{n-1}M_{n-1}$ for $n \in \mathbb{N}$ and let $M_{-1} = 0$.

Definition 4.2.1. Let $n \in \mathbb{N}_0$ and suppose we have already defined $\mathcal{J}_{n,\beta}$. Set

- $\mathcal{A}_{n,\beta} = \left(\mathcal{J}_{n,\beta} - (M_n - 1)\omega\right) \times C$,
- $\mathcal{B}_{n,\beta} = \left(\mathcal{J}_{n,\beta} + (M_n + 1)\omega\right) \times E$,
- $\mathcal{J}_{n+1,\beta} = \pi_{\mathbb{T}^d}\left(f_\beta^{M_n-1}(\mathcal{A}_{n,\beta}) \cap f_\beta^{-(M_n+1)}(\mathcal{B}_{n,\beta})\right)$.

Remark. It is obvious that $\mathcal{J}_{n+1,\beta} \subseteq \mathcal{J}_{n,\beta}$ ($n \in \mathbb{N}_0$). However, $\mathcal{J}_{n+1,\beta}$ might be empty even if $\mathcal{J}_{n,\beta} \neq \emptyset$.

Given $\theta \in \mathbb{T}^d$, $x \in X$, and $k \in \mathbb{Z}$, we may use the shorthand notation $\theta_k = \theta + k\omega$ and $x_k = f_{\beta,\theta}^k(x)$, where we keep the dependence on β implicit.

For fixed $N \in \mathbb{N}$, we will only consider such $\beta \in \mathscr{B}(0)$ with

$$f_{\beta,\theta_{-(M_n-1)}}^{M_n-1}(c^+) \geq f_{\beta,\theta_{M_n+1}}^{-(M_n+1)}(e^-)$$

for each $\theta \in \mathcal{J}_{n,\beta}$ and $0 \leq n \leq N-1$. We denote the set of these β by $\tilde{\mathscr{B}}(N)$ and set $\tilde{\mathscr{B}}(0) = \mathscr{B}(0)$.

Definition 4.2.2. For $n \in \mathbb{N}$, set $\mathcal{I}_n = \bigcup_{\beta\in\tilde{\mathscr{B}}(n)} \mathcal{J}_{n,\beta}$. We call $\mathcal{I}_n$ the *n-th critical region*. For $n \in \mathbb{N}_0$, set

- $\mathcal{Z}_n^- = \bigcup_{j=0}^n \bigcup_{l=-(M_j-2)}^0 \mathcal{I}_j + l\omega$ and $\mathcal{Z}_n^+ = \bigcup_{j=0}^n \bigcup_{l=0}^{M_j} \mathcal{I}_j + l\omega$;
- $\mathcal{X}_n^- = \bigcup_{j=0}^n \bigcup_{l=-(M_j-1)}^0 \mathcal{I}_j + l\omega$ and $\mathcal{X}_n^+ = \bigcup_{j=0}^n \bigcup_{l=1}^{M_j+1} \mathcal{I}_j + l\omega$.

Moreover, let $\mathcal{I}_{-1} = \mathcal{I}_0$ and $\mathcal{Z}_{-1}^-, \mathcal{Z}_{-1}^+, \mathcal{X}_{-1}^-, \mathcal{X}_{-1}^+ = \emptyset$.

In order to be able to control an orbit, we do not want it to visit the critical regions too often. We therefore need to assume that the critical regions are small enough. This idea is formalised in the following inductive assumptions

$(\mathcal{F}1)_n$ $\mathcal{I}_j \cap \bigcup_{k=1}^{2K_jM_j} \mathcal{I}_j + k\omega = \emptyset$ for $j = 0, \dots, n$;

$(\mathcal{F}2)_n$ $\left(\mathcal{I}_j - (M_j - 1)\omega \cup \mathcal{I}_j + (M_j + 1)\omega\right) \bigcap \left(\mathcal{X}_{j-1}^- \cup \mathcal{X}_{j-1}^+\right) = \emptyset$ for $j = 0, \dots, n$;

where $n \in \mathbb{N}_0$. If both $(\mathcal{F}1)_n$ and $(\mathcal{F}2)_n$ are verified, we simply say $(\mathcal{F})_n$ holds true. For notational reasons it is convenient to define $(\mathcal{F})_{-1}$ to be equivalent to $(\mathcal{F})_0$.

For $\theta \in \mathbb{T}^d$, we denote by $\mathcal{L}_m$ and $\mathcal{R}_m$ in $\mathbb{N}_0 \cup \{\infty\}$ the smallest numbers l and r with $\theta_l \in \mathcal{I}_m$ and $\theta_{-r} \in \mathcal{I}_m + \omega$, respectively.

Lemma 4.2.3 (cf. [Jäg09b, Lemma 3.4]). *Let $n \in \mathbb{N}_0$, $\beta \in \tilde{\mathscr{B}}(n)$ and assume $(\mathcal{F})_{n-1}$ is verified. Suppose f_β satisfies $(\mathcal{A}4)$, $(\mathcal{A}5)$ and $(\mathcal{A}5')$ and assume*

$$\begin{cases} x \in C \\ \theta \notin \mathcal{Z}^-_{n-1}. \end{cases} \tag{$(\mathcal{B}1)_n$}$$

Furthermore, let $0 < \mathcal{L}^{(1)} < \ldots < \mathcal{L}^{(N)} = \mathcal{L}_n$ be all those times $m \leq \mathcal{L}_n$ for which $\theta + m\omega \in \mathcal{I}_{n-1}$. Then $(\theta_{\mathcal{L}^{(i)}+M_{n-1}+2}, x_{\mathcal{L}^{(i)}+M_{n-1}+2})$ satisfies $(\mathcal{B}1)_n$ for each $i = 1, \ldots, N-1$ and the following implication holds

$$x_k \notin C \Rightarrow \theta_k \in \mathcal{X}^+_{n-1} \text{ and } x_k \in [e^-, c^-] \quad (k = 1, \ldots, \mathcal{L}_n). \tag{$(C1)_n$}$$

Analogously backwards: Instead of $(\mathcal{B}1)_n$, assume

$$\begin{cases} x \in E \\ \theta \notin \mathcal{Z}^+_{n-1} \end{cases} \tag{$(\mathcal{B}2)_n$}$$

and let $0 < \mathcal{R}^{(1)} < \ldots < \mathcal{R}^{(N)} = \mathcal{R}_n$ be all those times $m \leq \mathcal{R}_n$ for which $\theta - m\omega \in \mathcal{I}_{n-1} + \omega$. Then $(\theta_{-\mathcal{R}^{(i)}-M_{n-1}-1}, x_{-\mathcal{R}^{(i)}-M_{n-1}-1})$ satisfies $(\mathcal{B}2)_n$ for each $i = 1, \ldots, N-1$ and the following implication holds

$$x_{-k} \notin E \Rightarrow \theta_{-k} \in \mathcal{X}^-_{n-1} \text{ and } x_{-k} \in [e^+, c^+] \quad (k = 1, \ldots, \mathcal{R}_n). \tag{$(C2)_n$}$$

Proof. We only consider the forward case; the other case is similar. Note that for $n = 0$ the statement is true due to $(\mathcal{A}5)$.

Suppose the statement holds for $n_0 \in \mathbb{N}_0$ and (θ, x) satisfies $(\mathcal{B}1)_{n_0+1}$. Without loss of generality we may assume that $N > 1$. Trivially, $(\mathcal{B}1)_{n_0+1}$ implies $(\mathcal{B}1)_{n_0}$ such that $x_k \notin C$ implies $\theta_k \in \mathcal{X}^+_{n_0-1}$ and $x_k \in [e^-, c^+]$ for $k \leq \mathcal{L}^{(1)}$. Notice that $(\mathcal{I}_{n_0} - (M_{n_0} - 1)\omega) \cap \mathcal{X}_{n_0-1} = \emptyset$ because of $(\mathcal{F}2)_{n_0}$. Hence, $\left(\theta_{\mathcal{L}^{(1)}-(M_{n_0}-1)}, x_{\mathcal{L}^{(1)}-(M_{n_0}-1)}\right) \in \mathcal{A}_{n_0,\beta}$ due to $(C1)_{n_0}$. For $\beta \in \tilde{\mathscr{B}}(n_0 + 1)$, we further have $f^{M_{n_0}-1}_{\beta,\theta_{\mathcal{L}^{(1)}-M_{n_0}+1}}(c^+) \geq f^{-(M_{n_0}+1)}_{\beta,\theta_{\mathcal{L}^{(1)}+M_{n_0}+1}}(e^-)$. If we had $x_{\mathcal{L}^{(1)}} \leq f^{-(M_{n_0}+1)}_{\beta,\theta_{\mathcal{L}^{(1)}+M_{n_0}+1}}(e^-)$, this would imply the existence of $y \in [x_{\mathcal{L}^{(1)}-M_{n_0}+1}, c^+] \subseteq [c^-, c^+]$ with $f^{M_{n_0}-1}_\beta(\theta_{\mathcal{L}^{(1)}-M_{n_0}+1}, y) \in f^{-(M_{n_0}+1)}_\beta\left(\left\{\theta_{\mathcal{L}^{(1)}+M_{n_0}+1}\right\} \times E\right)$ meaning that $\theta_{\mathcal{L}^{(1)}} \in \mathcal{J}_{n_0+1,\beta} \subseteq \mathcal{I}_{n_0+1}$, which contradicts the assumption that $N > 1$. Therefore, $x_{\mathcal{L}^{(1)}} \geq f^{-(M_{n_0}+1)}_{\beta,\theta_{\mathcal{L}^{(1)}+M_{n_0}+1}}(e^-)$. By $(\mathcal{A}4)$ and the monotonicity, we thus have $x_k \in [e^-, c^+]$ for $k = \mathcal{L}^{(1)}, \ldots, \mathcal{L}^{(1)} + M_{n_0} + 1$. Now, $x_{\mathcal{L}^{(1)}+M_{n_0}+1} \notin E$, since otherwise again $\theta_{\mathcal{L}^{(1)}} \in \mathcal{J}_{n_0+1,\beta} \subseteq \mathcal{I}_{n_0+1}$, by definition of $\mathcal{J}_{n_0+1,\beta}$. $(\mathcal{A}5)$ and $(\mathcal{F}2)_{n_0}$ hence yield $x_{\mathcal{L}^{(1)}+M_{n_0}+2} \in C$. By $(\mathcal{F})_{n_0}$, we get that $(\mathcal{I}_{n_0} + (M_{n_0} + 2)\omega) \cap \mathcal{Z}^-_{n_0} = \emptyset$. Thus, $\left(\theta_{\mathcal{L}^{(1)}+M_{n_0}+2}, x_{\mathcal{L}^{(1)}+M_{n_0}+2}\right)$ verifies $(\mathcal{B}1)_{n_0+1}$. The statement follows by induction. □

Remark. If $(\mathcal{F}2)_n$ holds true for some $n \in \mathbb{N}_0$, then each $(\theta, x) \in \mathcal{A}_{n,\beta}$ satisfies $(\mathcal{B}1)_n$. If further $(\mathcal{F}1)_{n-1}$ holds, then we have $\mathcal{L}_n = M_n - 1$ for such (θ, x) since $M_n \leq 2K_{n-1}M_{n-1}$ for $n \in \mathbb{N}$ and $\mathcal{I}_n \subseteq \mathcal{I}_{n-1}$. Likewise, all $(\theta, x) \in \mathcal{B}_{n,\beta}$ satisfy $(\mathcal{B}2)_n$ and it holds $\mathcal{R}_n = M_n$.

Corollary 4.2.4. *Let $n \in \mathbb{N}_0$, $\beta \in \tilde{\mathscr{B}}(n)$ and assume $(\mathcal{F})_{n-1}$ as well as $(\mathcal{F}2)_n$ are verified. Suppose f_β satisfies $(\mathcal{A}4)$, $(\mathcal{A}5)$, and $(\mathcal{A}5')$. Then*

$$f_\beta^{M_n-1}(\mathcal{A}_{n,\beta}) \subseteq \mathcal{I}_n \times C \quad \text{and} \quad f_\beta^{-M_n}(\mathcal{B}_{n,\beta}) \subseteq (\mathcal{I}_n + \omega) \times E.$$

Proof. Note that $\mathcal{I}_n \cap \mathcal{X}^+_{n-1} = \emptyset$ and $\mathcal{I}_n + \omega \cap \mathcal{X}^-_{n-1} = \emptyset$, because of $(\mathcal{F}1)_{n-1}$. By means of the previous remark, the inclusions follow from $(C1)_n$ and $(C2)_n$ in Lemma 4.2.3. □

Corollary 4.2.5 (cf. [Jäg09b, Corollary 3.7])**.** *Let $N > n \in \mathbb{N}_0$, $\beta \in \tilde{\mathscr{B}}(N)$ and assume $(\mathcal{F}1)_{N-1}$ as well as $(\mathcal{F}2)_N$ are verified. Suppose f_β satisfies $(\mathcal{A}4)$, $(\mathcal{A}5)$, and $(\mathcal{A}5')$. Then*

$$f_\beta^{M_N-M_n}(\mathcal{A}_{N,\beta}) \subseteq (\mathcal{I}_n - (M_n-1)\omega) \times (c^-, c^+] \subseteq \mathcal{A}_{n,\beta};$$
$$f_\beta^{-M_N+M_n}(\mathcal{B}_{N,\beta}) \subseteq (\mathcal{I}_n + (M_n+1)\omega) \times [e^-, e^+) \subseteq \mathcal{B}_{n,\beta}.$$

Proof. Since $\mathcal{I}_{n+1} - (M_n-1)\omega \cap \mathcal{X}^+_n = \emptyset$, Lemma 4.2.3 yields $f_\beta^{M_{n+1}-M_n}(\mathcal{A}_{n+1,\beta}) \subseteq \mathcal{A}_{n,\beta}$. Observe that the proof of Lemma 4.2.3 actually yields the slightly stronger inclusion $f_\beta^{M_{n+1}-M_n}(\mathcal{A}_{n+1,\beta}) \subseteq (\mathcal{I}_n - (M_n-1)\omega) \times (c^-, c^+]$. Now, the first result follows by induction. The other relation follows similarly. □

By means of the next statement, we can estimate the amount of time spent in the contracting and in the expanding region. For $n, N \in \mathbb{N}$ set

$$\mathcal{P}_n^N(\theta, x) = \#\{l \in [n, N-1] \cap \mathbb{N}_0 \colon x_l \in C \text{ and } \theta_l \notin \mathcal{I}_0\}$$
$$\mathcal{Q}_n^N(\theta, x) = \#\{l \in [n, N-1] \cap \mathbb{N}_0 \colon x_{-l} \in E \text{ and } \theta_{-l} \notin \mathcal{I}_0 + \omega\}.$$

Set

$$b_0 = 1, \quad b_n = \left(1 - \frac{1}{K_{n-1}}\right) b_{n-1} \ (n \in \mathbb{N}).$$

Lemma 4.2.6 (cf. [Jäg09b, Lemma 3.8])**.** *Let $n \in \mathbb{N}_0$, $\beta \in \tilde{\mathscr{B}}(n)$ and assume $(\mathcal{F})_{n-1}$ holds true. Suppose f_β satisfies $(\mathcal{A}4)$, $(\mathcal{A}5)$, and $(\mathcal{A}5')$. Furthermore, assume (θ, x) verifies $(\mathcal{B}1)_n$ and let $0 < \mathcal{L}^{(1)} < \ldots < \mathcal{L}^{(N)} = \mathcal{L}_n$ be as in Lemma 4.2.3. Then, for each $i = 1, \ldots, N$, we have*

$$\mathcal{P}_k^{\mathcal{L}^{(i)}}(\theta, x) \geq b_n(\mathcal{L}^{(i)} - k) \quad (k = 0, \ldots, \mathcal{L}^{(i)}).$$

Analogously backwards: Instead of $(\mathcal{B}1)_n$, assume (θ, x) verifies $(\mathcal{B}2)_n$ and let $0 < \mathcal{R}^{(1)} < \ldots < \mathcal{R}^{(N)} = \mathcal{R}_n$ be as in Lemma 4.2.3. Then, for each $i = 1, \ldots, N$, we have

$$\mathcal{Q}_k^{\mathcal{R}^{(i)}}(\theta, x) \geq b_n(\mathcal{R}^{(i)} - k) \quad (k = 0, \ldots, \mathcal{R}^{(i)}).$$

Remark. For merely proving the existence of an SNA, it suffices to have the lower bound for $\mathcal{P}_k^{\mathcal{L}_n}(\theta, x)$ only. Nevertheless, the estimates for $\mathcal{P}_k^{\mathcal{L}^{(i)}}(\theta, x)$ are needed for the geometric analysis of the SNA in Chapter 5.

Proof. We consider the first inequality, the second one follows similarly. For $n = 0$, the statement follows from $(\mathcal{A}5)$.

Assume the statement is true for $n = n_0 \in \mathbb{N}_0$ and assume (θ, x) verifies $(\mathcal{B}1)_{n_0+1}$. Due to Lemma 4.2.3, we have that $(\theta_{\mathcal{L}^{(i)}+M_{n_0}+2}, x_{\mathcal{L}^{(i)}+M_{n_0}+2})$ satisfies $(\mathcal{B}1)_{n_0+1}$ for $i = 1, \ldots, N-1$. By the induction hypothesis, we thus get the desired estimate for $P_k^{\mathcal{L}^{(i+1)}}(\theta, x)$ as long as either $k \in [\mathcal{L}^{(i)} + M_{n_0} + 2, \mathcal{L}^{(i+1)}]$ for some $1 \leq i \leq N-1$ or $k \in [0, \mathcal{L}^{(1)}]$ and $i = 0$.

Moreover by $(\mathcal{F}1)_{n_0}$, we have

$$\mathcal{L}^{(i+1)} - \mathcal{L}^{(i)} \geq 2K_{n_0} M_{n_0}. \tag{4.2.1}$$

Hence, for all $k \in [\mathcal{L}^{(i)}, \mathcal{L}^{(i)} + M_{n_0} + 1]$ we get

$$\begin{aligned}\mathcal{P}_k^{\mathcal{L}^{(i+1)}}(\theta, x) &\geq \mathcal{P}_{\mathcal{L}^{(i)}+M_{n_0}+2}^{\mathcal{L}^{(i+1)}}(\theta, x) \geq b_{n_0}(\mathcal{L}^{(i+1)} - (\mathcal{L}^{(i)} + M_{n_0} + 2)) \\ &\geq b_{n_0}(\mathcal{L}^{(i+1)} - \mathcal{L}^{(i)} - 2M_{n_0}) \overset{(4.2.1)}{\geq} b_{n_0+1}(\mathcal{L}^{(i+1)} - \mathcal{L}^{(i)}) \geq b_{n_0+1}(\mathcal{L}^{(i+1)} - k).\end{aligned}$$

Altogether, with $j = \min\{l = 1, \ldots, N \colon \mathcal{L}^{(l)} \geq k\}$ and $N \geq i \geq j$ we therefore have

$$\begin{aligned}\mathcal{P}_k^{\mathcal{L}^{(i)}}(\theta, x) &= \mathcal{P}_k^{\mathcal{L}^{(j)}}(\theta, x) + \sum_{l=j}^{i-1} \mathcal{P}_{\mathcal{L}^{(l)}}^{\mathcal{L}^{(l+1)}}(\theta, x) \geq b_{n_0+1}\left(\mathcal{L}^{(j)} - k + \sum_{l=j}^{i-1} \mathcal{L}^{(l+1)} - \mathcal{L}^{(l)}\right) \\ &= b_{n_0+1}(\mathcal{L}^{(i)} - k).\end{aligned}$$

□

Corollary 4.2.7 (cf. [Jäg09b, Corollary 3.9]). *Let $n \in \mathbb{N}_0$, $\beta \in \tilde{\mathscr{B}}(n)$ and assume $(\mathcal{F})_{n-1}$ as well as $(\mathcal{F}2)_n$ are verified. Suppose f_β satisfies $(\mathcal{A}1)$–$(\mathcal{A}5)$ and let $(\theta, x) \in f_\beta^{M_n}(\mathcal{A}_{n,\beta})$. Then*

$$\partial_x f_{\beta,\theta}^{-k}(x) \geq \left(\alpha_c^{b_n} \alpha_u^{1-b_n}\right)^{-k} \quad (0 \leq k \leq M_n).$$

Analogously, let $(\theta, x) \in f_\beta^{-M_n}(\mathcal{B}_{n,\beta})$. Then

$$\partial_x f_{\beta,\theta}^{k}(x) \geq \left(\alpha_e^{b_n} \alpha_l^{1-b_n}\right)^{k} \quad (0 \leq k \leq M_n).$$

Proof. We only treat the first case, the second one follows analogously. Without loss of generality we may assume $k \geq 1$. First of all, note that for θ and x as in the hypothesis we have

$$\begin{aligned}0 < \left(\partial_x f_{\beta,\theta-k\omega}^{k}\right)(f_{\beta,\theta}^{-k}(x)) &\leq \alpha_c^{\mathcal{P}_{M_n-k}^{M_n-1}\left(\theta-M_n\omega, f_{\beta,\theta}^{-M_n}(x)\right)+1} \cdot \alpha_u^{k-1-\mathcal{P}_{M_n-k}^{M_n-1}\left(\theta-M_n\omega, f_{\beta,\theta}^{-M_n}(x)\right)} \\ &\leq \alpha_c^{k\cdot b_n} \cdot \alpha_u^{k\cdot(1-b_n)},\end{aligned}$$

where we used Corollary 4.2.4 and Lemma 4.2.6. Now, since $f_{\beta,\theta-k\omega}^{k}(f_{\beta,\theta}^{-k}(x)) = x$, we get by the chain rule that

$$1 = \partial_x f_{\beta,\theta-k\omega}^{k}(f_{\beta,\theta}^{-k}(x)) \leq \partial_x f_{\beta,\theta}^{-k}(x) \cdot \left(\alpha_c^{b_n} \alpha_u^{1-b_n}\right)^{k}.$$

The statement follows immediately. □

Define

- $b = \lim_{n\to\infty} b_n$,
- $\alpha_- = \alpha_c^b \alpha_u^{1-b}$,
- $\alpha_+ = \alpha_e^b \alpha_l^{1-b}$.

Proposition 4.2.8 (cf. [Jäg09b, Proposition 3.10]). *Assume $\mathcal{I}_n \neq \emptyset$ for all $n \in \mathbb{N}_0$, $(\mathcal{F})_n$ is verified for all $n \in \mathbb{N}_0$, $\beta \in \bigcap_{n\in\mathbb{N}} \tilde{\mathscr{B}}(n) \neq \emptyset$, and f_β verifies $(\mathcal{A}1)$–$(\mathcal{A}5)$. Moreover, assume $\alpha_-^{-1}, \alpha_+ > 1$ and for all $n \in \mathbb{N}$ we have $f_\beta^{M_n}(\mathcal{A}_{n,\beta}) \cap f_\beta^{-M_n}(\mathcal{B}_{n,\beta}) \neq \emptyset$. Then there exists a sink-source orbit for f_β in $\mathbb{T}^d \times [e^-, c^+]$ and hence an SNA and an SNR. More precisely,*

$$\{(\theta, x) \in \mathbb{T}^d \times X \colon (\theta, x) \text{ is a sink-source orbit}\} \supseteq \bigcap_{n\in\mathbb{N}} \left(f_\beta^{M_n}(\mathcal{A}_{n,\beta}) \cap f_\beta^{-M_n}(\mathcal{B}_{n,\beta}) \right) \neq \emptyset.$$

Proof. It follows from Corollary 4.2.5 that the above intersection is non-empty. The rest is a consequence of Corollary 4.2.7 and the fact that $\alpha_-^{-1}, \alpha_+ > 1$ because for $(\theta, x) \in \bigcap_{n\in\mathbb{N}}(f_\beta^{M_n}(\mathcal{A}_{n,\beta}) \cap f_\beta^{-M_n}(\mathcal{B}_{n,\beta}))$, arbitrary $k \in \mathbb{N}_0$, and n such that $M_n \geq k$ we have

$$\partial_x f_{\beta,\theta}^{-k}(x) \geq \left(\alpha_c^{b_n} \alpha_u^{1-b_n}\right)^{-k} \geq \alpha_-^{-k}, \qquad \partial_x f_{\beta,\theta}^{k}(x) \geq \left(\alpha_e^{b_n} \alpha_l^{1-b_n}\right)^{k} \geq \alpha_+^{k}. \qquad \square$$

4.2.2. Geometric considerations

In this paragraph, we get an upper bound for the size of the n-th critical region $\mathcal{I}_n$. So far, we dealt with $\beta \in \tilde{\mathscr{B}}(n)$ in order to guarantee that the respective orbits stay in the strip $\mathbb{T}^d \times [e^-, c^+]$. Due to the monotonicity in β (provided by $(\mathcal{A}8)$), this amounts to only considering small enough β. On the other hand, $\tilde{\mathscr{B}}(n)$ also contains parameters β which are *too small* such that $\mathcal{J}_{n,\beta} = \emptyset$, which is not desirable either. In order to exclude these parameters as well, we define the set of *admissible parameters* up to order $n \in \mathbb{N}$ by

$$\begin{aligned} &\mathscr{B}(n) \\ &= \left\{\beta \in \tilde{\mathscr{B}}(n) \colon \mathcal{J}_{l,\beta} \neq \emptyset \text{ for } 0 \leq l \leq n\right\} \\ &= \left\{\beta \in \tilde{\mathscr{B}}(n) \colon \exists \theta \in \mathcal{J}_{l,\beta} \text{ such that } f_{\beta,\theta_{-(M_l-1)}}^{M_l-1}(c^-) \leq f_{\beta,\theta_{M_l+1}}^{-(M_l+1)}(e^+) \text{ where } 0 \leq l \leq n-1\right\}, \end{aligned}$$

where we assume $(M_l)_{l=0,\dots,n-1}$ to be given.

Proposition 4.2.9. *Suppose $(f_\beta)_{\beta\in[0,1]}$ satisfies $(\mathcal{A}6)$ and $(\mathcal{A}8)$ and let $\beta < \beta' \in \tilde{\mathscr{B}}(n)$ for $n \in \mathbb{N}_0$. Then*

$$\mathcal{J}_{n,\beta} \subseteq \mathcal{J}_{n,\beta'}. \tag{4.2.2}$$

In particular, this implies that $\mathscr{B}(n)$ is an interval.

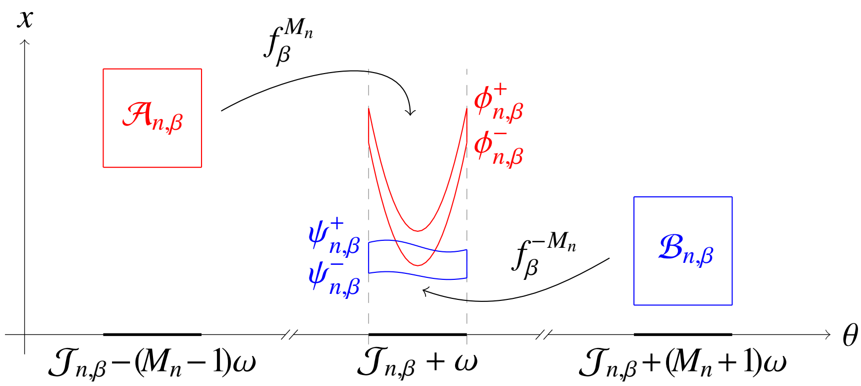

Figure 4.1. The geometric idea for proving the smallness of the critical regions.

Proof. For $n = 0$, (4.2.2) holds by ($\mathcal{A}6$). Assume (4.2.2) is true for some $n \in \mathbb{N}_0$. For $\beta \in \tilde{\mathscr{B}}(n+1)$, we know $\theta \in \mathcal{J}_{n+1,\beta}$ if and only if $0 \geq f^{M_n-1}_{\beta,\theta_{-(M_n-1)}}(c^-) - f^{-(M_n+1)}_{\beta,\theta_{M_n+1}}(e^+)$. Since $f_{(\cdot)}(\theta, x)$ is non-increasing, $f^{M_n-1}_{(\cdot),\theta_{-(M_n-1)}}(c^-) - f^{-(M_n+1)}_{(\cdot),\theta_{M_n+1}}(e^+)$ is non-increasing, too. Hence, $\theta \in \mathcal{J}_{n+1,\beta}$ implies $\theta \in \mathcal{J}_{n+1,\beta'}$. Now, the statement follows by induction. □

Up to now, we basically used monotonicity in β in order to investigate the set of admissible parameters. In order to guarantee that $\mathscr{B}(n)$ is non-empty and to control the size of the critical regions $\mathcal{I}_n$, we need subtler geometric information. The intuitive idea of the argument for the smallness of $\mathcal{I}_n$ can be seen by considering $\mathcal{J}_{1,\beta}$: As $f^j_\beta(\mathcal{A}_{0,\beta})$ stays in the contracting region for $j = 0, \ldots, M_0 - 1$, the iterates of $\mathcal{A}_{0,\beta}$ become thinner and thinner horizontal strips (assume $d = 1$ for simplicity) with each step of the iteration until they meet $\mathcal{I}_0 \times C$. Likewise, $f_\beta^{-M_0}(\mathcal{B}_{0,\beta})$ is basically a thin horizontal strip. Iterating $f_\beta^{M_0-1}(\mathcal{A}_{0,\beta})$ once more deforms the previously horizontal strip to a thin strip around a parabola with second derivative at least s because of ($\mathcal{A}9$). This yields an upper bound for the size of $\mathcal{J}_{1,\beta}$ and hence, by considering all $\beta \in \mathscr{B}(1)$, an upper bound for $|\mathcal{I}_1|$.

The smallness of $\mathcal{I}_n$ for arbitrary n follows in a similar fashion, but we have to show that even though the iterates of $\mathcal{A}_{n,\beta}$ enter the expanding region from time to time, the overall effect of the iteration under f_β is still a contraction (see Figure 4.1).

In order to formalise this intuitive idea, we define the functions

$$\phi^\pm_{n,\beta}(\theta) = f^{M_n}_{\beta,\theta-M_n\omega}(c^\pm) \quad \text{and} \quad \psi^\pm_{n,\beta}(\theta) = f^{-M_n}_{\beta,\theta+M_n\omega}(e^\pm)$$

for $\theta \in \mathcal{J}_{n,\beta} + \omega$, $n \in \mathbb{N}_0$. Note that

$$f_\beta^{M_n}\left(\mathcal{A}_{n,\beta}\right) = \left\{(\theta, x) \in \left(\mathcal{J}_{n,\beta} + \omega\right) \times X \colon\ x \in [\phi^-_{n,\beta}(\theta), \phi^+_{n,\beta}(\theta)]\right\},$$
$$f_\beta^{-M_n}\left(\mathcal{B}_{n,\beta}\right) = \left\{(\theta, x) \in \left(\mathcal{J}_{n,\beta} + \omega\right) \times X \colon\ x \in [\psi^-_{n,\beta}(\theta), \psi^+_{n,\beta}(\theta)]\right\}$$

(cf. Figure 4.1). We introduce a shorthand notation for the following inductive assumptions.

$$\left.\begin{aligned} \mathscr{B}(n) \text{ is a non-empty and closed interval,} && (4.2.3)\\ \mathcal{J}_{n,\beta} \text{ is non-empty, closed, and convex for } \beta \in \mathscr{B}(n), && (4.2.4)\\ \phi^-_{n,\beta}(\theta) > \psi^+_{n,\beta}(\theta) \text{ for each } \theta \in \partial\mathcal{J}_{n,\beta} + \omega \text{ and } \beta \in \mathscr{B}(n), && (4.2.5)\\ \exists \beta_-(n+1) \in \mathcal{B}(n)\,\&\,\exists!\theta^n_- \in \mathcal{J}_{n,\beta_-(n+1)} + \omega\colon\ \phi^-_{n,\beta_-(n+1)}(\theta^n_-) = \psi^+_{n,\beta_-(n+1)}(\theta^n_-), && (4.2.6)\\ \exists \beta_+(n+1) \in \mathcal{B}(n)\,\&\,\exists!\theta^n_+ \in \mathcal{J}_{n,\beta_+(n+1)} + \omega\colon\ \phi^+_{n,\beta_+(n+1)}(\theta^n_+) = \psi^-_{n,\beta_+(n+1)}(\theta^n_+). && (4.2.7) \end{aligned}\right\} (\mathcal{I})_n$$

Moreover, set

- $H^\phi_n = \sup_{\beta\in\mathscr{B}(n),\, \theta\in\mathcal{J}_{n,\beta}} |\phi^+_{n,\beta}(\theta) - \phi^-_{n,\beta}(\theta)|$,
- $H^\psi_n = \sup_{\beta\in\mathscr{B}(n),\, \theta\in\mathcal{J}_{n,\beta}} |\psi^+_{n,\beta}(\theta) - \psi^-_{n,\beta}(\theta)|$,
- $\nu^\tau_n = \inf_{\substack{\beta\in\mathscr{B}(n),\, \theta\in\mathcal{J}_{n,\beta}\\ \vartheta\in\mathbb{S}^{d-1}}} \partial^2_\vartheta\phi^\tau_{n,\beta}(\theta) - \partial^2_\vartheta\psi^{-\tau}_{n,\beta}(\theta)$ $(\tau \in \{-,+\})$.

Lemma 4.2.10. *Assume $(\mathcal{I})_n$ holds for some $n \in \mathbb{N}_0$. Then $\mathscr{B}(n+1)$ is non-empty. Further, assume $(\mathcal{F}1)_n$ and $(\mathcal{F}2)_{n+1}$ hold true, let $\beta \in \mathscr{B}(n+1)$ and suppose f_β satisfies $(\mathcal{A}4)$–$(\mathcal{A}6)$, $(\mathcal{A}5')$, and $(\mathcal{A}8)$. If $\nu^\pm_n, \nu^\pm_{n+1} > 0$, then*

- *$(\mathcal{I})_{n+1}$ holds,*
- *$|\mathcal{I}_{n+1}| \leq \sqrt{8}\sqrt{\frac{H^\phi_n + H^\psi_n}{\nu^-_n}}$.*

Proof. Note that $\emptyset \neq \mathscr{B}(n+1) = [\beta_-(n+1), \beta_+(n+1)]$ by (4.2.6), (4.2.7) as well as Proposition 4.2.9 and $(\mathcal{A}8)$.

$\mathcal{J}_{n+1,\beta}$ is a sublevel set of $\phi^-_{n,\beta} - \psi^+_{n,\beta}$ and hence, it is closed. Given two points $\theta_1, \theta_2 \in \mathcal{J}_{n+1,\beta}$, denote by $[\theta_1, \theta_2] \subseteq \mathcal{J}_{n,\beta}$ the line joining the two points. As $\partial^2_{\vartheta'}\phi^-_{n,\beta}(\theta) - \partial^2_{\vartheta'}\psi^+_{n,\beta}(\theta) \geq \nu^-_n > 0$ (with ϑ' the unit vector in direction of $\theta_2 - \theta_1$), we have $\phi^-_{n,\beta} - \psi^+_{n,\beta} \leq 0$ on $[\theta_1, \theta_2]$ and thus convexity of $\mathcal{J}_{n+1,\beta}$.

By applying Corollary 4.2.5, we see that $[\phi^-_{n+1,\beta}(\theta), \phi^+_{n+1,\beta}(\theta)] \subseteq (\phi^-_{n,\beta}(\theta), \phi^+_{n,\beta}(\theta)]$ as well as $[\psi^-_{n+1,\beta}(\theta), \psi^+_{n+1,\beta}(\theta)] \subseteq [\psi^-_{n,\beta}(\theta), \psi^+_{n,\beta}(\theta))$ for all $\theta \in \mathcal{J}_{n+1,\beta} + \omega, \beta \in \mathscr{B}(n+1)$. This ensures $\phi^-_{n+1,\beta} > \psi^+_{n+1,\beta}$ on $\partial\mathcal{J}_{n+1,\beta}$ and guarantees that

$$\begin{aligned} \beta_+(n+2) &= \min\left\{\beta \in \mathscr{B}(n+1) \,\middle|\, \exists\theta \in \mathcal{J}_{n+1,\beta} + \omega\colon\ \phi^+_{n+1,\beta}(\theta) \leq \psi^-_{n+1,\beta}(\theta)\right\}\\ &= \min\left\{\beta \in \mathscr{B}(n+1) \,\middle|\, \exists\theta \in \mathcal{J}_{n+1,\beta} + \omega\colon\ \phi^+_{n+1,\beta}(\theta) = \psi^-_{n+1,\beta}(\theta)\right\} \end{aligned}$$

as well as

$$\beta_-(n+2) = \max\left\{\beta \in \mathscr{B}(n+1) \,\middle|\, \beta < \beta_+(n+2),\ \forall\theta \in \mathcal{J}_{n+1,\beta} + \omega\colon\ \phi^-_{n+1,\beta}(\theta) \geq \psi^+_{n+1,\beta}(\theta)\right\}$$

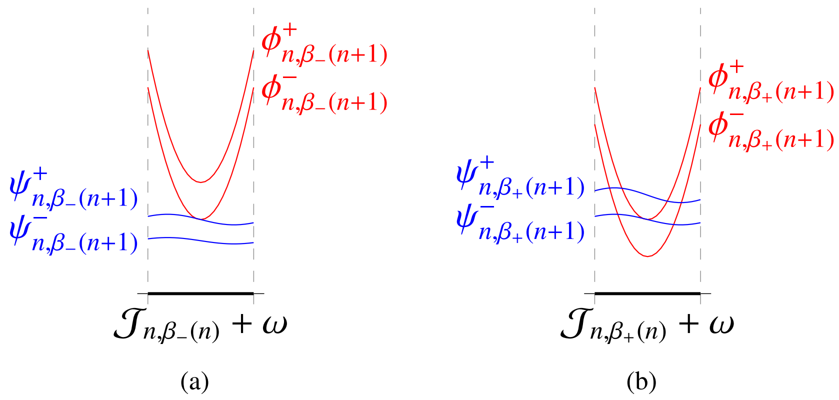

Figure 4.2. (a) $\mathcal{J}_{n+1,\beta_-(n+1)}$ is degenerate. (b) $\beta_+(n+1)$ is the largest parameter such that $\mathcal{J}_{n+1,\beta_+(n+1)}$ is connected.

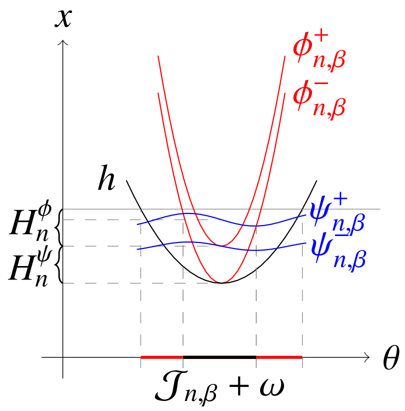

Figure 4.3. An upper bound for the size of $\mathcal{J}_{n,\beta}$. Note that h touches $\phi^-_{n,\beta}$ in this picture for illustrative reasons only.

are well-defined (see Figure 4.2). Using $v^\pm_{n+1} > 0$, we get the uniqueness of the tangent points of $\phi^-_{n+1,\beta_-(n+1)}$ and $\psi^+_{n+1,\beta_-(n+1)}$ as well as of $\phi^+_{n+1,\beta_+(n+1)}$ and $\psi^-_{n+1,\beta_+(n+1)}$ and conclude $(\mathcal{I})_{n+1}$.

Note that $\phi^-_{n,\beta} - \psi^+_{n,\beta} \geq -\left(H^\phi_n + H^\psi_n\right)$ and furthermore, $\partial^2_\vartheta\left(\phi^-_{n,\beta} - \psi^+_{n,\beta}\right) \geq v^-_n$. Suppose $\phi^-_{n,\beta} - \psi^+_{n,\beta}$ assumes its minimum at $\theta_0 \in \mathcal{J}_{n,\beta} + \omega$ and define $h\colon \theta \mapsto v^-_n/2 \cdot |\theta - \theta_0|^2 - \left(H^\phi_n + H^\psi_n\right)$. As $\phi^-_{n,\beta}(\theta) - \psi^+_{n,\beta}(\theta) \leq 0$ if and only if $\theta \in \mathcal{J}_{n+1,\beta} + \omega$, we necessarily have that $h(\theta) \leq 0$ for all $\theta \in \mathcal{J}_{n+1,\beta} + \omega$ such that an upper bound for the size of $\mathcal{J}_{n+1,\beta}$ is given by the distance of the zeros of h (cf. Figure 4.3). □

Lemma 4.2.11. *Suppose* $(\mathcal{A}5)$–$(\mathcal{A}8)$ *and* $(\mathcal{F}1)_0$ *are verified and* $v^\pm_0 > 0$. *Then* $(\mathcal{I})_0$ *holds true.*

Proof. Recall that by means of $(\mathcal{A}7)$ and $(\mathcal{A}8)$, we defined $\mathcal{B}(0) \subseteq [0,1]$ to be a non-empty and closed interval such that $\mathcal{J}_{0,\beta}$ is non-empty for each $\beta \in \mathscr{B}(0)$. Further in $(\mathcal{A}6)$, we assumed $\mathcal{J}_{0,\beta} \subseteq \mathbb{T}^d$ to be closed and convex. Moreover, with $(\mathcal{A}5)$ and $(\mathcal{F}1)_0$ we have (4.2.5) for $n = 0$. Finally and similarly as in the proof of Lemma 4.2.10, we can define $\beta_-(1)$ and $\beta_+(1)$ such that (4.2.6) and (4.2.7) are verified for $n = 0$, where the uniqueness of the tangent points follows from $v_0^\pm > 0$. □

We next provide estimates for the quantities used in Lemma 4.2.10.

Lemma 4.2.12 (cf. [Jäg09b, Lemma 3.13]). *Let $n \in \mathbb{N}_0$, $\beta \in \tilde{\mathscr{B}}(n)$ and assume $(\mathcal{F})_{n-1}$, $(\mathcal{F}2)_n$, and $(\mathcal{I})_n$ hold true. Suppose f_β verifies $(\mathcal{A}1)$–$(\mathcal{A}5)$. Then*

$$H^\phi_{n,\beta} \leq \left(\alpha_c^{b_n}\alpha_u^{1-b_n}\right)^{M_n} \cdot |C| \qquad \text{and} \qquad H^\psi_{n,\beta} \leq \left(\alpha_e^{b_n}\alpha_l^{1-b_n}\right)^{-M_n} \cdot |E|.$$

Proof. Apply Corollary 4.2.7. □

The following statement is, from a technical point of view, the core part of this chapter. It provides us with a positive lower bound for $v_n^\pm$ and thereby ensures that we can apply Lemma 4.2.10. The idea is to show that the second derivative of $\phi^\pm_{n,\beta}(\theta) - \psi^\mp_{n,\beta}(\theta) = f^{M_n}_{\beta,\theta-M_n\omega}(c^\pm) - f^{-M_n}_{\beta,\theta+M_n\omega}(e^\mp)$ in direction ϑ differs from $\left(\partial^2_\vartheta f_{\beta,\theta-\omega}\right)\left(f^{M_n-1}_{\beta,\theta-M_n\omega}(c^\pm)\right)$ by a remainder term only, whose supremum goes to zero with increasing expansion and contraction rates. Since $(\mathcal{A}9)$ provides us with a lower bound s for the second derivative of f_β with respect to the base coordinates in every direction, this proves the claim.

Lemma 4.2.13. *Let $n \in \mathbb{N}_0$, $\beta \in \tilde{\mathscr{B}}(n)$ and assume $(\mathcal{F})_{n-1}$ and $(\mathcal{F}2)_n$ hold true. Suppose f_β satisfies $(\mathcal{A}1)$–$(\mathcal{A}5)$ as well as $(\mathcal{A}9)$–$(\mathcal{A}15)$. Let there be $p \geq \sqrt{2}$ and $\alpha > 1$ such that*

$$\alpha_c^{-1} = \alpha_e = \alpha^{2/p}, \qquad \alpha_l^{-1} = \alpha_u = \alpha^p$$

and assume $b_n > 5p^2/(2 + 5p^2)$. Then

$$v_n^\pm \geq s - S^2\mathfrak{c} \cdot \alpha^{-(2b_n/p - 5(1-b_n)p)},$$

where $\mathfrak{c} = \mathfrak{c}(\alpha, b_n) > 0$ can be chosen to be monotonously decreasing in both arguments.

Proof. For reasons of readability, we omit the index β in the following. Let us consider $\frac{\partial^2}{\partial\vartheta^2}\phi_n^\pm(\theta)$ ($\theta \in \mathcal{J}_n + \omega$ and $\vartheta \in \mathbb{S}^{d-1}$). Set $\theta_0 = \theta - M_n\omega$ and $x_k^\pm = f^k_{\theta_0}(c^\pm)$. Since there is no difference in the treatment of x_k^+ and x_k^-, we skip the superscript in the following. It is worth mentioning that due to Lemma 4.2.3 x_k never leaves $[e^-, c^+]$ for $k = 0, \ldots, M_n - 1$. Now

$$\begin{aligned}\frac{\partial}{\partial\vartheta}\phi_n^\pm(\theta) &= \partial_\vartheta f^{M_n}_{\theta_0}(c^\pm) = \left(\partial_\vartheta f_{\theta_{M_n-1}}\right)(x_{M_n-1}) + \left(\partial_x f_{\theta_{M_n-1}}\right)(x_{M_n-1}) \cdot \partial_\vartheta f^{M_n-1}_{\theta_0}(c^\pm) \\ &= \ldots = \left(\partial_\vartheta f_{\theta_{M_n-1}}\right)(x_{M_n-1}) + \sum_{k=0}^{M_n-2}\left(\partial_\vartheta f_{\theta_k}\right)(x_k) \cdot \left(\partial_x f^{M_n-k-1}_{\theta_{k+1}}\right)(x_{k+1}),\end{aligned} \tag{4.2.8}$$

where we used

$$\left(\partial_x f_{\theta_{k+1}}^{M_n-k-1}\right)(x_{k+1}) = \prod_{j=k+1}^{M_n-1} \left(\partial_x f_{\theta_j}\right)(x_j) \quad (k = -1, 0, \dots, M_n - 1). \tag{4.2.9}$$

Differentiating once more gives

$$\begin{aligned}\frac{\partial^2}{\partial\vartheta^2}\phi_n^{\pm}(\theta) = {} & \left(\partial_\vartheta^2 f_{\theta_{M_n-1}}\right)(x_{M_n-1}) + \left(\partial_x\partial_\vartheta f_{\theta_{M_n-1}}\right)(x_{M_n-1})\cdot\partial_\vartheta f_{\theta_0}^{M_n-1}(c^\pm) \\ & + \sum_{k=0}^{M_n-2} (\partial_\vartheta f_{\theta_k})(x_k)\cdot\partial_\vartheta\left(\partial_x f_{\theta_{k+1}}^{M_n-k-1}\right)(x_{k+1}) + \left[\partial_\vartheta\left(\partial_\vartheta f_{\theta_k}\right)(x_k)\right](\partial_x f_{\theta_{k+1}}^{M_n-k-1})(x_{k+1}).\end{aligned}$$

Further,

$$\begin{aligned}\partial_\vartheta\left(\partial_x f_{\theta_{k+1}}^{M_n-k-1}\right)(x_{k+1}) &= \partial_\vartheta \prod_{j=k+1}^{M_n-1}\left(\partial_x f_{\theta_j}\right)(x_j) = \sum_{l=k+1}^{M_n-1}\partial_\vartheta\left(\partial_x f_{\theta_l}\right)(x_l)\prod_{\substack{j=k+1\\ j\neq l}}^{M_n-1}\left(\partial_x f_{\theta_j}\right)(x_j) \\ &= \sum_{l=k+1}^{M_n-1}\left[\left(\partial_\vartheta\partial_x f_{\theta_l}\right)(x_l) + \left(\partial_x^2 f_{\theta_l}\right)(x_l)\cdot\partial_\vartheta f_{\theta_0}^{l}(c^\pm)\right]\prod_{\substack{j=k+1\\ j\neq l}}^{M_n-1}\left(\partial_x f_{\theta_j}\right)(x_j)\end{aligned}$$

and

$$\partial_\vartheta\left(\partial_\vartheta f_{\theta_k}\right)(x_k) = \left(\partial_\vartheta^2 f_{\theta_k}\right)(x_k) + \left(\partial_x\partial_\vartheta f_{\theta_k}\right)(x_k)\cdot\partial_\vartheta f_{\theta_0}^{k}(c^\pm).$$

Altogether, we have

$$\begin{aligned}&\frac{\partial^2}{\partial\vartheta^2}\phi_n^{\pm}(\theta) \\ &= \left(\partial_\vartheta^2 f_{\theta_{M_n-1}}\right)(x_{M_n-1}) + \left(\partial_x\partial_\vartheta f_{\theta_{M_n-1}}\right)(x_{M_n-1})\cdot\partial_\vartheta f_{\theta_0}^{M_n-1}(c^\pm) \\ &+ \sum_{k=0}^{M_n-2}\left(\partial_\vartheta f_{\theta_k}\right)(x_k)\left(\sum_{l=k+1}^{M_n-1}\left[\left(\partial_\vartheta\partial_x f_{\theta_l}\right)(x_l) + \left(\partial_x^2 f_{\theta_l}\right)(x_l)\cdot\partial_\vartheta f_{\theta_0}^{l}(c^\pm)\right]\prod_{\substack{j=k+1\\ j\neq l}}^{M_n-1}\left(\partial_x f_{\theta_j}\right)(x_j)\right) \\ &+ \left[\left(\partial_\vartheta^2 f_{\theta_k}\right)(x_k) + \left(\partial_x\partial_\vartheta f_{\theta_k}\right)(x_k)\cdot\partial_\vartheta f_{\theta_0}^{k}(c^\pm)\right]\left(\partial_x f_{\theta_{k+1}}^{M_n-k-1}\right)(x_{k+1}).\end{aligned} \tag{4.2.10}$$

It is our goal to show that the long times spent in the contracting region keep the derivatives small, such that $\left(\partial_\vartheta^2 f_{\theta_{M_n-1}}\right)(x_{M_n-1})$ becomes the leading term. The part which is the hardest to control is

$$\sum_{k=0}^{M_n-2}\left(\partial_\vartheta f_{\theta_k}\right)(x_k)\sum_{l=k+1}^{M_n-1}\left(\partial_x^2 f_{\theta_l}\right)(x_l)\cdot\partial_\vartheta f_{\theta_0}^{l}(c^\pm)\prod_{\substack{j=k+1\\ j\neq l}}^{M_n-1}\left(\partial_x f_{\theta_j}\right)(x_j),$$

with $\partial_\vartheta f^l_{\theta_0}(c^\pm) = \sum_{m=0}^{l-1} (\partial_\vartheta f_{\theta_m})(x_m) \cdot \left(\partial_x f^{l-m-1}_{\theta_{m+1}}\right)(x_{m+1})$ as in equation (4.2.8). Using ($\mathscr{A}$10), we see that it is bounded from above by

$$S^2 \sum_{k=0}^{M_n-2} \sum_{l=k+1}^{M_n-1} \sum_{m=0}^{l-1} \left|\left(\partial_x^2 f_{\theta_l}\right)(x_l)\right| \left(\partial_x f^{l-m-1}_{\theta_{m+1}}\right)(x_{m+1}) \prod_{\substack{j=k+1\\ j\neq l}}^{M_n-1} \left(\partial_x f_{\theta_j}\right)(x_j). \tag{4.2.11}$$

If $m \leq k$, then

$$\begin{aligned}
&\left|\left(\partial_x^2 f_{\theta_l}\right)(x_l)\right| \left(\partial_x f^{l-m-1}_{\theta_{m+1}}\right)(x_{m+1}) \prod_{\substack{j=k+1\\ j\neq l}}^{M_n-1} \left(\partial_x f_{\theta_j}\right)(x_j) \\
&= \left|\left(\partial_x^2 f_{\theta_l}\right)(x_l)\right| \prod_{\substack{j=m+1\\ j\neq l}}^{M_n-1} \left(\partial_x f_{\theta_j}\right)(x_j) \cdot \prod_{j=k+1}^{l-1} \left(\partial_x f_{\theta_j}\right)(x_j) \\
&\leq \left|\left(\partial_x^2 f_{\theta_l}\right)(x_l)\right| \prod_{\substack{j=m+1\\ j\neq l\\ x_j\in C}}^{M_n-1} \alpha_c \cdot \prod_{\substack{j=m+1\\ j\neq l\\ x_j\notin C}}^{M_n-1} \alpha_u \prod_{\substack{j=k+1\\ x_j\notin C}}^{l-1} \alpha_u \leq \left|\left(\partial_x^2 f_{\theta_l}\right)(x_l)\right| \prod_{\substack{j=m+1\\ j\neq l\\ x_j\in C}}^{M_n-1} \alpha_c \cdot \prod_{\substack{j=m+1\\ j\neq l\\ x_j\notin C}}^{M_n-1} \alpha_u^2 \\
&\leq \alpha_c^{b_n(M_n-m-1)} \alpha_u^{2(1-b_n)(M_n-m-1)} = \alpha_1^{-(M_n-m-1)},
\end{aligned}$$

where we used Lemma 4.2.6 and ($\mathscr{A}$13) in the last estimate and where we set $\alpha_1 = \alpha_c^{-b_n} \alpha_u^{-2(1-b_n)} = \alpha^{-2(p(1-b_n)-b_n/p)}$. For $m > k$ we get an analogous result with m replaced by k. Hence, (4.2.11) is bounded by

$$\begin{aligned}
&S^2 \sum_{k=0}^{M_n-2} \sum_{l=k+1}^{M_n-1} \left(\sum_{m=0}^{k} \alpha_1^{-(M_n-1-m)} + \sum_{m=k+1}^{l-1} \alpha_1^{-(M_n-1-k)} \right) \\
&\leq S^2 \sum_{k=0}^{M_n-2} \sum_{l=k+1}^{M_n-1} \left(\alpha_1^{-(M_n-1-k)} \sum_{m=0}^{k} \alpha_1^{-m} + \alpha_1^{-(M_n-1-k)} \sum_{m=k+1}^{l-1} 1 \right) \\
&\leq S^2 \sum_{k=0}^{M_n-2} \left(\alpha_1^{-(M_n-1-k)} \frac{1}{1-\alpha_1^{-1}} (M_n-k-1) + \alpha_1^{-(M_n-1-k)} \sum_{l=k+1}^{M_n-1} (l-k-1) \right) \\
&\leq S^2 \frac{2}{1-\alpha_1^{-1}} \sum_{k=0}^{M_n-2} \alpha_1^{-(M_n-1-k)} (M_n-k-1)^2 \leq S^2 \frac{2\alpha_1}{\alpha_1 - 1} \sum_{l=1}^{M_n-1} l^2 \alpha_1^{-l} \\
&\leq S^2 \tilde{c}(\alpha_1) \cdot \alpha_1^{-1},
\end{aligned}$$

where $\tilde{c}(\alpha) = \frac{2\alpha}{\alpha-1} \sum_{l=1}^{\infty} l^2 \alpha^{-l+1}$ for each $\alpha > 1$.[4] Note that $\tilde{c}$ is monotonously decreasing in α. The other addends of (4.2.10) can be treated in a similar fashion, which eventually gives

$$\frac{\partial^2}{\partial\vartheta^2} \phi_n^\pm(\theta) \geq \left(\partial_\vartheta^2 f_{\theta_{M_n-1}}\right)(x_{M_n-1}) - 5S^2 \tilde{c}(\alpha_1) \cdot \alpha_1^{-1} \overset{(\mathscr{A}9)}{\geq} s - 5S^2 \tilde{c}(\alpha_1) \cdot \alpha_1^{-1}.$$

[4] Notice that $\alpha_1 > 1$, since $b_n > (5p^2)/(2+5p^2) > p^2/(p^2+1)$.

Now, let us consider $\frac{\partial^2}{\partial\vartheta^2}\psi_n^{\pm}(\theta) = f_{\theta+M_n\omega}^{-M_n}(e^{\pm})$ for $\vartheta \in \mathbb{S}^{d-1}$. We proceed similarly as before but this time we consider the map f^{-1} instead of f and set $\theta_k = \theta_0 - k\omega$ (with $\theta_0 = \theta + M_n\omega$ for $\theta \in \mathcal{J}_n + \omega$) as well as $x_k^{\pm} = f_{\theta_0}^{-k}(e^{\pm})$. As before, we skip the superscript of x_k^+ and x_k^- in the following. First, let us provide some simple computations which yield estimates on the derivatives of the inverse map. Observe that

$$\partial_x f_\theta^{-1}(x) = \frac{1}{(\partial_x f_{\theta-\omega})\left(f_\theta^{-1}(x)\right)} \quad \left(\text{hence, } 0 < \partial_x f_\theta^{-1}(x) < \alpha_e^{-1} \ (x \in E, \theta \notin \mathcal{I}_0 + \omega)\right),$$

$$\partial_\vartheta f_\theta^{-1}(x) = -\frac{(\partial_\vartheta f_{\theta-\omega})\left(f_\theta^{-1}(x)\right)}{(\partial_x f_{\theta-\omega})\left(f_\theta^{-1}(x)\right)} = -(\partial_\vartheta f_{\theta-\omega})\left(f_\theta^{-1}(x)\right) \cdot \partial_x f_\theta^{-1}(x),$$

within $\{(\theta, x) \in f(\mathbb{T}^d \times X) \colon Df(f^{-1}(\theta, x)) \text{ is non-singular}\} \supseteq f(\mathbb{T}^d \times [e^-, c^+]) \cap \mathbb{T}^d \times [e^-, c^+]$. Hence,

$$\partial_x^2 f_\theta^{-1}(x) = -\frac{(\partial_x^2 f_{\theta-\omega})\left(f_\theta^{-1}(x)\right) \cdot \partial_x f_\theta^{-1}(x)}{\left[(\partial_x f_{\theta-\omega})\left(f_\theta^{-1}(x)\right)\right]^2} = -(\partial_x^2 f_{\theta-\omega})\left(f_\theta^{-1}(x)\right) \cdot \left(\partial_x f_\theta^{-1}(x)\right)^3$$

such that $\left|\partial_x^2 f_\theta^{-1}(x)\right| \leq \alpha_u^2 \alpha_l^{-3}$ and

$$\begin{aligned}\partial_\vartheta \partial_x f_\theta^{-1}(x) &= -\frac{(\partial_\vartheta \partial_x f_{\theta-\omega})\left(f_\theta^{-1}(x)\right) + (\partial_x^2 f_{\theta-\omega})\left(f_\theta^{-1}(x)\right) \cdot \partial_\vartheta f_\theta^{-1}(x)}{\left[(\partial_x f_{\theta-\omega})\left(f_\theta^{-1}(x)\right)\right]^2} \\ &= -(\partial_\vartheta \partial_x f_{\theta-\omega})\left(f_\theta^{-1}(x)\right) \cdot \left(\partial_x f_\theta^{-1}(x)\right)^2 - (\partial_\vartheta f_{\theta-\omega})\left(f_\theta^{-1}(x)\right) \cdot \partial_x^2 f_\theta^{-1}(x)\end{aligned}$$

and thus $\left|\partial_\vartheta \partial_x f_\theta^{-1}(x)\right| \leq 2S\alpha_u^2 \alpha_l^{-3}$ for $(\theta, x) \in f(\mathbb{T}^d \times [e^-, c^+]) \cap \mathbb{T}^d \times [e^-, c^+]$. Finally,

$$\begin{aligned}&\partial_\vartheta^2 f_\theta^{-1}(x) \\ &= -(\partial_\vartheta^2 f_{\theta-\omega})\left(f_\theta^{-1}(x)\right) \cdot \partial_x f_\theta^{-1}(x) - (\partial_x \partial_\vartheta f_{\theta-\omega})\left(f_\theta^{-1}(x)\right) \cdot \partial_\vartheta f_\theta^{-1}(x) \partial_x f_\theta^{-1}(x) \\ &\quad - (\partial_\vartheta f_{\theta-\omega})\left(f_\theta^{-1}(x)\right) \cdot \partial_\vartheta \partial_x f_\theta^{-1}(x) \\ &= -(\partial_\vartheta^2 f_{\theta-\omega})\left(f_\theta^{-1}(x)\right) \cdot \partial_x f_\theta^{-1}(x) - 2(\partial_\vartheta f_{\theta-\omega})\left(f_\theta^{-1}(x)\right) \cdot \partial_\vartheta \partial_x f_\theta^{-1}(x) \\ &\quad - \left((\partial_\vartheta f_{\theta-\omega})\left(f_\theta^{-1}(x)\right)\right)^2 \cdot \partial_x^2 f_\theta^{-1}(x).\end{aligned} \tag{4.2.12}$$

As in the forward case, we get

$$\begin{aligned}&\frac{\partial^2}{\partial\vartheta^2}\psi_n^{\pm}(\theta) \\ &= \sum_{k=0}^{M_n-1} \left(\partial_\vartheta f_{\theta_k}^{-1}\right)(x_k) \left(\sum_{l=k+1}^{M_n-1} \left[\left(\partial_\vartheta \partial_x f_{\theta_l}^{-1}\right)(x_l) + \left(\partial_x^2 f_{\theta_l}^{-1}\right)(x_l) \cdot \partial_\vartheta f_{\theta_0}^{-l}(e^{\pm})\right] \prod_{\substack{j=k+1 \\ j \neq l}}^{M_n-1} \left(\partial_x f_{\theta_j}^{-1}\right)(x_j) \right) \\ &\quad + \left[\left(\partial_\vartheta^2 f_{\theta_k}^{-1}\right)(x_k) + \left(\partial_\vartheta \partial_x f_{\theta_k}^{-1}\right)(x_k) \cdot \partial_\vartheta f_{\theta_0}^{-k}(e^{\pm})\right] \left(\partial_x f_{\theta_{k+1}}^{-(M_n-k-1)}\right)(x_{k+1}).\end{aligned} \tag{4.2.13}$$

As we consider iterates of the inverse map, we want to show that the long times spent in the expanding region keep all the derivatives small. With $\partial_\vartheta f_{\theta_0}^{-l}(e^\pm) = \sum_{m=0}^{l-1} \left(\partial_\vartheta f_{\theta_m}^{-1}\right)(x_m) \cdot \left(\partial_x f_{\theta_{m+1}}^{-(l-m-1)}\right)(x_{m+1})$, the term which is the hardest to control in (4.2.13) is

$$\begin{aligned}
&\sum_{k=0}^{M_n-1} \left(\partial_\vartheta f_{\theta_k}^{-1}\right)(x_k) \sum_{l=k+1}^{M_n-1} \left(\partial_x^2 f_{\theta_l}^{-1}\right)(x_l)\, \partial_\vartheta f_{\theta_0}^{-l}(e^\pm) \prod_{\substack{j=k+1\\ j\neq l}}^{M_n-1} \left(\partial_x f_{\theta_j}^{-1}\right)(x_j)\\
&= -\sum_{k=0}^{M_n-2} (\partial_\vartheta f_{\theta_{k+1}})\,(x_{k+1})\, \partial_x f_{\theta_k}^{-1}(x_k) \sum_{l=k+1}^{M_n-1} (\partial_x^2 f_{\theta_l}^{-1})(x_l)\\
&\cdot \sum_{m=0}^{l-1} (\partial_\vartheta f_{\theta_{m+1}})\,(x_{m+1})\, \partial_x f_{\theta_m}^{-1}(x) \left(\partial_x f_{\theta_{m+1}}^{-(l-m-1)}\right)(x_{m+1}) \cdot \prod_{\substack{j=k+1\\ j\neq l}}^{M_n-1} \left(\partial_x f_{\theta_j}^{-1}\right)(x_j).
\end{aligned}$$

An upper bound for this expression reads

$$S^2 \sum_{k=0}^{M_n-2} \sum_{l=k+1}^{M_n-1} \sum_{m=0}^{l-1} \left|(\partial_x^2 f_{\theta_l}^{-1})(x_l)\right| \cdot \prod_{n=m}^{l-1} \left(\partial_x f_{\theta_n}^{-1}\right)(x_n) \prod_{\substack{j=k\\ j\neq l}}^{M_n-1} \left(\partial_x f_{\theta_j}^{-1}\right)(x_j). \tag{4.2.14}$$

We deal similarly with (4.2.14) as we did with (4.2.11). Suppose $m > k$. Since $\left(\partial_x f_\theta^{-1}(x)\right)^2 < \alpha_l^{-2} < \alpha_u^2 \alpha_l^{-3}$, we get

$$\begin{aligned}
&\left|(\partial_x^2 f_{\theta_l}^{-1})(x_l)\right| \prod_{n=m}^{l-1} \left(\partial_x f_{\theta_n}^{-1}\right)(x_n) \prod_{\substack{j=k\\ j\neq l}}^{M_n-1} \left(\partial_x f_{\theta_j}^{-1}\right)(x_j)\\
&\leq \left|(\partial_x^2 f_{\theta_l}^{-1})(x_l)\right| \prod_{\substack{j=k\\ j\neq l\\ x_j\in E\wedge\theta_j\notin\mathcal{I}_0+\omega}}^{M_n-1} \left(\partial_x f_{\theta_j}^{-1}\right)(x_j) \prod_{\substack{j=k\\ j\neq l\\ x_j\notin E\vee\theta_j\in\mathcal{I}_0+\omega}}^{M_n-1} \left(\partial_x f_{\theta_n}^{-1}\right)^2(x_n)\\
&\overset{(\mathcal{A}14)}{\leq} \alpha_e^{-b_n(M_n-k)} \left(\alpha_u^2\alpha_l^{-3}\right)^{(1-b_n)(M_n-k)} \leq \alpha_2^{-(M_n-k)},
\end{aligned}$$

where $\alpha_2 = \alpha_e^{b_n}(\alpha_u^2\alpha_l^{-3})^{-(1-b_n)} = \alpha^{2b_n/p-5(1-b_n)p}$. For $m \leq k$, we get an analogous result. Hence, (4.2.14) is bounded by

$$\begin{aligned}
&S^2 \sum_{k=0}^{M_n-2} \sum_{l=k+1}^{M_n-1} \left(\alpha_2^{-(M_n-k)} \sum_{m=0}^{k} \alpha_2^{-m} + \alpha_2^{-(M_n-k)} \sum_{m=k+1}^{l-1} 1\right)\\
&\leq S^2 \frac{2\alpha_2}{\alpha_2-1} \sum_{k=0}^{M_n-2} \alpha_2^{-(M_n-k)}(M_n-k)^2 \leq S^2 \tilde{c}(\alpha_2)\cdot\alpha_2^{-2},
\end{aligned}$$

with $\tilde{c}$ as in the forward case.[5] Nevertheless, notice that

$$\begin{aligned}
&\left|\sum_{k=0}^{M_n-1}\left(\partial_\vartheta^2 f_{\theta_k}^{-1}\right)(x_k)\left(\partial_x f_{\theta_{k+1}}^{-(M_n-k-1)}\right)(x_{k+1})\right| \\
&\leq \sum_{k=0}^{M_n-1} S^2\left(\partial_x f_{\theta_k}^{-(M_n-k)}\right)(x_k) + 2S\left|\partial_\vartheta\partial_x f_{\theta_k}^{-1}(x_k)\right|\left(\partial_x f_{\theta_{k+1}}^{-(M_n-k-1)}\right)(x_{k+1}) \\
&\quad + S^2\partial_x^2 f_{\theta_k}^{-1}(x_k)\left(\partial_x f_{\theta_{k+1}}^{-(M_n-k-1)}\right)(x_{k+1}) \leq 3S^2\tilde{c}(\alpha_2)\alpha_2^{-1},
\end{aligned}$$

where we used (4.2.12) in the first step. Altogether, we eventually get

$$\frac{\partial^2}{\partial\vartheta^2}\psi^\pm(\theta) \leq 6S^2\tilde{c}(\alpha_2)\cdot\alpha_2^{-1}.$$

Setting $\mathfrak{c}(\alpha, b_n) = 6\tilde{c}(\alpha^{2b_n/p-5(1-b_n)p}) + 5\tilde{c}(\alpha^{2b_n/p-2(1-b_n)p})$ yields the desired estimate. □

4.2.3. Existence of a sink-source orbit

In Section 4.2.1, we proved the existence of a sink source orbit for f_β provided there are strictly increasing sequences $(M_n)_{n\in\mathbb{N}_0}, (K_n)_{n\in\mathbb{N}_0} \in \mathbb{N}^{\mathbb{N}_0}$ such that the inductively defined critical regions $\mathcal{I}_n$ are non-empty and satisfy $(\mathcal{F}1)_n, (\mathcal{F}2)_n$. By means of the geometric considerations of the last section, we are now able to show that such sequences $(M_n)_{n\in\mathbb{N}_0}, (K_n)_{n\in\mathbb{N}_0}$ actually do exist. This finishes the proof of Theorem 4.1.1.

As a matter of fact, we prove that the critical regions satisfy a slightly stronger version of $(\mathcal{F}1)_n$, that is, we will show

$$d\left(\mathcal{I}_j, \bigcup_{k=1}^{2K_jM_j}\mathcal{I}_j + k\omega\right) > \varepsilon_j \geq |\mathcal{I}_j| \qquad (\mathcal{F}1)'_n$$

for $j = 0, \ldots, n$ and $n \in \mathbb{N}_0$ where $\varepsilon_0 = \mathscr{C}(2K_0M_0)^{-\eta}/3$ (with $\mathscr{C}$ and η as in Definition 3.1.1) and $\varepsilon_j = \tilde{\mathscr{C}}\tilde{\alpha}^{-M_{j-1}}$ for all $j \in \mathbb{N}$ and some positive constants $\tilde{\mathscr{C}}$ and $\tilde{\alpha} > 1$.

Lemma 4.2.14 (cf. [Jäg09b, Lemma 3.16]). *Let $n \in \mathbb{N}$ and suppose we are given K_ℓ, M_ℓ ($\ell = 0, \ldots, n-1$) such that $(\mathcal{F}1)'_{n-1}$ and $(\mathcal{F}2)_{n-1}$ hold true. If $\sum_{\ell=0}^{n-1} 1/K_\ell \leq \frac{1}{6}$, then there exists $M_n \in [K_{n-1}M_{n-1}, 2K_{n-1}M_{n-1}]$ such that $(\mathcal{F}2)_n$ is verified.*

Proof. Let $j = 0, \ldots, n-1$. Then,

$$\mathcal{I}_n - (M_n - 1)\omega \cap \bigcup_{l=-(M_j-1)}^{M_j+1}\mathcal{I}_j + l\omega \neq \emptyset$$

[5] Note that $\alpha_2 > 1$, since $b > 1 - 2/(2 + 5p^2)$.

implies

$$\mathcal{I}_j - (M_n - 1)\omega \cap \mathcal{I}_j + l\omega \neq \emptyset,$$

for some $l \in \{-M_j + 1, -M_j + 2, \dots, M_j + 1\}$. By $(\mathcal{F}1)'_{n-1}$,

$$\#\left\{q \in [K_{n-1}M_{n-1}, 2K_{n-1}M_{n-1}] \cap \mathbb{N} \,\middle|\, \mathcal{I}_j - (q-1)\omega \cap \mathcal{I}_j + l\omega \neq \emptyset\right\} \leq \frac{K_{n-1}M_{n-1}}{2K_jM_j}.$$

Hence,

$$\#\left\{q \in [K_{n-1}M_{n-1}, 2K_{n-1}M_{n-1}] \cap \mathbb{N} \,\middle|\, \mathcal{I}_j - (q-1)\omega \cap \bigcup_{l=-(M_j-1)}^{M_j+1} \mathcal{I}_j + l\omega \neq \emptyset\right\}$$
$$\leq (2M_j + 1)\frac{K_{n-1}M_{n-1}}{2K_jM_j}.$$

For the number of $q \in \{K_{n-1}M_{n-1}, K_{n-1}M_{n-1} + 1, \dots, 2K_{n-1}M_{n-1}\}$ with $\mathcal{I}_j + (q+1)\omega \cap \bigcup_{l=-(M_j-1)}^{M_j+1} \mathcal{I}_j + l\omega \neq \emptyset$, we get the same upper bound. Therefore, the number of integers q in $[K_{n-1}M_{n-1}, 2K_{n-1}M_{n-1}]$ with

$$(\mathcal{I}_n - (q-1)\omega \cup \mathcal{I}_n + (q+1)\omega) \cap \bigcup_{j=0}^{n-1} \bigcup_{l=-(M_j-1)}^{M_j+1} \mathcal{I}_j + l\omega \neq \emptyset$$

is bounded by $2(K_{n-1}M_{n-1}) \sum_{j=0}^{n-1}(2M_j+1)/(2K_jM_j) \leq 3(K_{n-1}M_{n-1}) \sum_{j=0}^{n-1} 1/K_j$. Thus, if $\sum_{j=0}^{n-1} 1/K_j \leq 1/6$, there is $M_n \in [K_{n-1}M_{n-1}, 2K_{n-1}M_{n-1}] \cap \mathbb{N}$ such that $(\mathcal{F}2)_n$ holds. □

Given $\alpha > 1$ and $b_1 = 1 - 1/K_0$, set

$$\nu = s - \mathfrak{c}\left(\alpha, b_1^2\right) S^2 \alpha^{-(2b_1^2/p - 5(1-b_1^2)p)},$$

where $\mathfrak{c}(\alpha, b_1^2)$ is as in Lemma 4.2.13. Theorem 4.1.1 follows from the next statement.

Theorem 4.2.15. *Suppose $\omega \in \mathbb{T}^d$ is Diophantine of type $(\mathscr{C}, \eta)$, $X \subseteq \mathbb{R}$ is some non-degenerate interval and $(f_\beta)_{\beta\in[0,1]} \in \mathcal{P}_\omega(X)$ satisfies $(\mathcal{A}1)$–$(\mathcal{A}15)$. Let there be $p \geq \sqrt{2}$ and $\alpha > 1$ with*

$$\alpha_c^{-1} = \alpha_e \geq \alpha^{2/p}, \qquad \alpha_l^{-1} = \alpha_u \leq \alpha^p.$$

Further, assume $3|\mathcal{I}_0| < \mathscr{C}(2K_0M_0)^{-\eta}$ for some integers M_0 not smaller than 2 and K_0 such that $2b_1^2/p - 5(1-b_1^2)p > 0$ and assume $\nu > 0$ as well as $\alpha > \alpha_0$, where $\alpha_0 = \alpha_0(\nu, K_0, M_0, p, |C|, |E|, \eta, \mathscr{C})$. Then there is $\beta_c \in [0,1]$ such that f_{β_c} has a sink-source orbit in $\mathbb{T}^d \times [e^-, c^+]$, and hence an SNA and an SNR.

Remark. We can choose α_0 to depend non-increasing in ν and non-decreasing in $|C|$ and $|E|$. Further, note that by the assumptions on K_0 we necessarily have $K_0 > 6$.

Proof. $(\mathcal{F}1)'_n$ is necessarily satisfied if $3|\mathcal{I}_\ell| \leq 3\varepsilon_\ell < d(k\omega, 0)$ $(k = 1, \ldots, 2K_\ell M_\ell)$ for $\ell = 0, \ldots, n$. Note that since ω is Diophantine of type $(\mathscr{C}, \eta)$, we have

$$\mathscr{C}(2K_n M_n)^{-\eta} < d(k\omega, 0) \tag{4.2.15}$$

for $k = 1, \ldots, 2K_n M_n$. Thus, $(\mathcal{F}1)'_0$ holds by the assumptions. Therefore, Lemma 4.2.14 yields the existence of $M_1 \in [K_0 M_0, 2K_0 M_0]$ such that $(\mathcal{F}2)_1$ holds. Lemma 4.2.13 gives $\nu_0^\pm, \nu_1^\pm \geq \nu > 0$ such that Lemma 4.2.11 gives $(\mathcal{I})_0$ and hence Lemma 4.2.10 yields $(\mathcal{I})_1$. By means of Lemma 4.2.10 together with Lemma 4.2.12, we further get

$$|\mathcal{I}_1| \leq \tilde{\mathscr{C}} \alpha_c^{M_0/2} \leq \tilde{\mathscr{C}} \tilde{\alpha}^{-M_0}$$

where $\tilde{\mathscr{C}} = \sqrt{8 \frac{|C|+|E|}{\nu}}$ and $\tilde{\alpha} = \alpha^{b_1^2/p - (1-b_1^2)p/2}$.

We set $(K_n)_{n \in \mathbb{N}_0} = (K_0 \kappa^n)_{n \in \mathbb{N}_0}$ for some $\kappa \in \mathbb{N}$ large enough to guarantee that $b_n > b > b_1^2$ and $\sum_{n=0}^\infty 1/K_n < 1/6$. Then, since $M_1 \in [K_0 M_0, 2K_0 M_0]$, we have for $n = 1$ that (4.2.15) is bounded from below by

$$\mathscr{C}(2K_1 M_1)^{-\eta} \geq \frac{\mathscr{C}}{\left(4\kappa K_0^2 M_0\right)^\eta}.$$

Therefore, $(\mathcal{F}1)'_1$ is verified if $\tilde{\alpha}$–and hence α–is large enough.

Let n be bigger than 1. Suppose $\nu_{n-1}^\pm \geq \nu$ and $(\mathcal{I})_{n-1}$, $(\mathcal{F}1)'_{n-1}$, and $(\mathcal{F}2)_{n-1}$ hold true with $K_j = K_0 \kappa^j$ and $M_j \in [K_{j-1} M_{j-1}, 2K_{j-1} M_{j-1}]$ for $j = 1, \ldots, n-1$. As for $n = 1$, Lemma 4.2.14 yields $M_n \in [M_{n-1} K_{n-1}, 2M_{n-1} K_{n-1}]$ such that $(\mathcal{F}2)_n$ holds. Now, $\nu_n^\pm \geq \nu$ and $(\mathcal{I})_n$ follow similarly as in the case $n = 1$. By means of Lemma 4.2.10 and Lemma 4.2.12, we get

$$\begin{aligned} 3\,|\mathcal{I}_n| &\leq 3\tilde{\mathscr{C}} \alpha_c^{b_{n-1} M_{n-1}/2} \alpha_u^{(1-b_{n-1}) M_{n-1}/2} \leq 3\tilde{\mathscr{C}} \alpha^{\left(-b_1^2/p + p(1-b_1^2)/2\right) M_{n-1}} = 3\tilde{\mathscr{C}} \tilde{\alpha}^{-M_{n-1}} \\ &\leq \frac{\mathscr{C}}{\left(4\kappa^{2n-1} K_0^2 M_{n-1}\right)^\eta} \end{aligned}$$

and thereby $(\mathcal{F}1)'_n$ for $\beta \in \mathscr{B}(n)$, where the last inequality holds for all $n \in \mathbb{N}$ if $\tilde{\alpha}$–and hence α–is large enough.

By induction, we thus see that there is a sequence $(M_n)_{n \in \mathbb{N}_0}$ such that $(\mathcal{I})_n$, $(\mathcal{F}1)'_n$, and $(\mathcal{F}2)_n$ hold true for all $n \in \mathbb{N}_0$. Applying Proposition 4.2.8 finishes the proof. □

We are now in a position to define the set $\mathcal{U}_\omega(X)$.

Definition 4.2.16. For Diophantine $\omega \in \mathbb{T}^d$, $\mathcal{U}_\omega(X)$ denotes the collection of all skew-product families in $\mathcal{P}_\omega(X)$ that undergo a saddle-node bifurcation and verify the assumptions of Theorem 4.2.15.

It is obvious that each family in $\mathcal{U}_\omega(X)$ goes through a non-smooth bifurcation. Due to Theorem 2.1.24, it is further clear that $\mathcal{U}_\omega(X)$ has non-empty interior. Finally, from the proof of the last theorem, we immediately get the following corollary.

Proposition 4.2.17. *Suppose $\hat{f} \in \mathcal{U}_\omega(X)$ and denote the critical parameter by β_c. Then there are integers $\kappa, M_0 \geq 2$, $K_0 \in \mathbb{N}$ and $(M_n)_{n\in\mathbb{N}}$ with $M_n \in [K_{n-1}M_{n-1}, 2K_{n-1}M_{n-1}]$ for all $n \in \mathbb{N}$, where $K_n = K_0\kappa^n$, so that the following holds.*

$\mathcal{I}_n \supseteq \mathcal{J}_{n,\beta_c} \neq \emptyset$ and $(\mathcal{F}1)'_n$ as well as $(\mathcal{F}2)_n$ are verified for all $n \in \mathbb{N}_0$. In particular, $|\mathcal{I}_n| \leq \varepsilon_n$ where $\varepsilon_n = \tilde{\mathscr{C}}\tilde{\alpha}^{-M_{n-1}}$ for some $\tilde{\alpha} > 1$ and all $n \in \mathbb{N}$.

5. Geometry of strange attractors

The theme of this chapter is a close look at the topological and geometrical properties of the SNA's that appear as the outcome of saddle-node bifurcations of families in a subset $\mathcal{V}_\omega(X) \subseteq \mathcal{U}_\omega(X)$ (still, with non-empty interior). To that end, we aim at an understanding of the SNA as the limit of so-called *iterated bounding lines*. This point of view yields certain Lipschitz conditions for the SNA which are the basis for our geometric examination.

These Lipschitz conditions and their discussion on an intuitive level is contained in the next section. In order to not obstruct the view on the principle ideas of the geometric analysis, their rather technical proof, which relies on the combinatorial ideas developed in Section 4.2.1, is postponed to the fourth and last section of this chapter.

In the second paragraph, we compute the Hausdorff dimension of the SNA. The box-counting dimension and the minimality of the maximal invariant set are dealt with in the third paragraph.

5.1. Iterated bounding lines

Let us start by defining the set $\mathcal{V}_\omega(X)$.

Definition 5.1.1. Given Diophantine $\omega \in \mathbb{T}^d$ and a non-degenerate interval $X \subseteq \mathbb{R}$, set

$$\mathcal{V}_\omega(X) = \left\{ \hat{f} \in \mathcal{U}_\omega(X) \colon b = \lim_{n\to\infty} b_n = \prod_{\ell=0}^{\infty}(1 - 1/K_\ell) > \sqrt{(p^2+1)/(p^2+2)} \right\}.$$

Recall that the limit b in the above definition is exactly the one given in Section 4.2.1 and observe that the assumption on b basically means a tighter smallness condition for $\mathcal{I}_0$ or a larger lower bound α_0 for the contraction/expansion constant α than the ones assumed for merely proving the occurrence of a non-smooth bifurcation in Theorem 4.2.15.

Our analysis of the structure of the SNA appearing in parameter families $\hat{f} \in \mathcal{V}_\omega(X)$ hinges on the fact that the SNA can be approximated by the images of the curve $\mathbb{T}^d \times \{c^+\}$ under successive iteration of the map f_{β_c}.

In the following, we restrict our attention to the critical parameter β_c and thus the map f_{β_c}. We therefore suppress the parameter from the notation. In this spirit, f always denotes a map in

$$\mathcal{V} = \left\{ f \in \mathcal{F}_\omega(X) \colon \text{there exists } \hat{f} \in \mathcal{V}_\omega(X) \text{ with } f = f_{\beta_c} \right\},$$

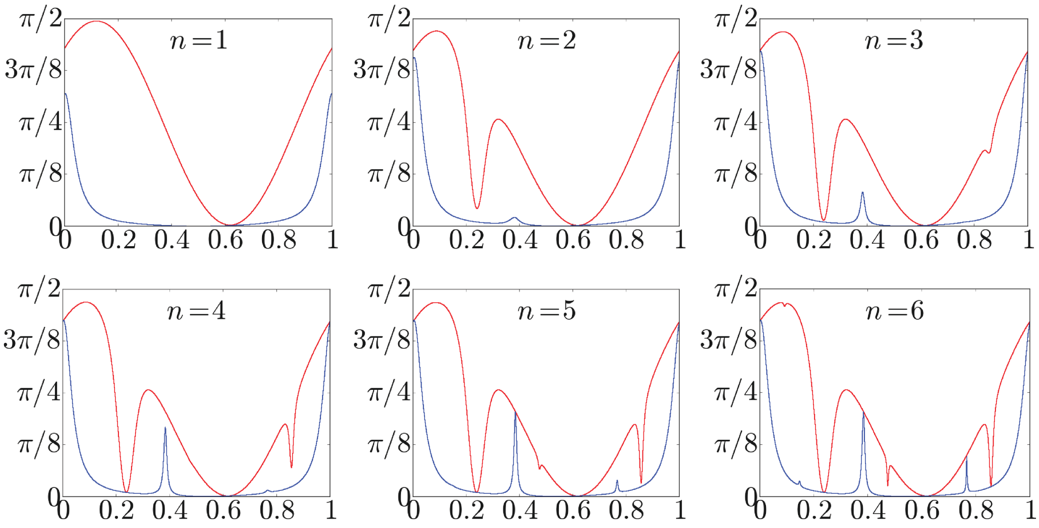

Figure 5.1. The first 6 iterated upper and lower bounding lines ϕ_n^+ (red) and ϕ_n^- (blue), respectively, of the family $(*)$ (see Section 4.1) for $a = 40$ at $\beta = 0.7729846$ with ω the golden mean.

where we keep the dependence on ω and X implicit. As in Section 3.2, we let

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(\Gamma)$$

be the maximal f-invariant set inside Γ. We denote by ϕ^- and ϕ^+ its boundary graphs, that is, $\phi^-(\theta) = \inf \Lambda(\theta)$ and $\phi^+(\theta) = \sup \Lambda(\theta)$ (cf. Definition 2.1.1). Now given $\theta \in \mathbb{T}^d$, let

$$\phi_n^+(\theta) = f^n_{\theta-n\omega}(c^+) = f_{\theta-\omega} \circ \ldots \circ f_{\theta-n\omega}(c^+);$$
$$\phi_n^-(\theta) = f^{-n}_{\theta+n\omega}(e^-) = f^{-1}_{\theta+\omega} \circ \ldots \circ f^{-1}_{\theta+n\omega}(e^-).$$

We call ϕ_n^+ the *n-th iterated upper bounding line* and ϕ_n^- the *n-th iterated lower bounding line*. The monotonicity of the fibre maps and $(\mathcal{A}4)$ yield that $(\phi_n^+)_{n\in\mathbb{N}}$ and $(\phi_n^-)_{n\in\mathbb{N}}$ are monotonously decreasing and increasing, respectively. Moreover, it is easy to see that $[\phi_n^-, \phi_n^+] = \bigcap_{k=-n}^n f^n(\Gamma)$. As a consequence, it is immediate that

$$\phi^+(\theta) = \lim_{n\to\infty} \phi_n^+(\theta) \quad \text{and} \quad \phi^-(\theta) = \lim_{n\to\infty} \phi_n^-(\theta).$$

Thus, in order to draw conclusions on the structure of the boundary graphs, it is natural to study the iterated bounding lines first. Figure 5.1 shows the first 6 iterated bounding lines for the critical parameter in the example family $(*)$ from Section 4.1 with ω the golden mean and parameters $a = 40$ and $\beta_c \approx 0.7729846$. These pictures reveal a very characteristic pattern. Let us look carefully at the evolution of ϕ_n^+.

For $n = 1$, we see that a first peak exists in the vicinity of $\theta = \omega$, that is, above the set $\mathcal{I}_0 + \omega$ (cf. ($\mathscr{A}$5)). After a second iteration, the image of this peak appears as a second peak in the vicinity of 2ω while outside this new peak the graph seems–more or less–unchanged. The second peak is not as pronounced as the first peak yet since the strong expansion close to the zero line (due to ($\mathscr{A}$2)) enlarged the tiny gap between $\phi_1^+(\omega)$ and $\phi_1^-(\omega)$. However, after one more iteration, the second peak is *stabilised*, that is, its shape is essentially fixed for higher iterations. It is also important to observe that the graph outside this peak has not changed apart from a small neighbourhood of 3ω in the step from $n = 2$ to $n = 3$. Furthermore, note that the second peak is of much smaller size than the first one.

Though the third peak around 3ω is already hardly visible at $n = 3$, it clearly stabilises until $n = 6$ and the graph only changes close to 4ω and 5ω along this stabilisation. Altogether, this motivates the following qualitative claim.

> *ϕ_{n+1}^+ differs from ϕ_n^+ only in smaller and smaller neighbourhoods of those peaks around $j\omega$ (for $j = 1, \ldots, n+1$) which are not stabilised yet after n iterations.*

The point is that every peak eventually stabilises in those θ which are not hit by peaks that appear at higher iterations. Moreover, the measure of these future peaks tends to zero. As ϕ_j^+ is Lipschitz-continuous with a Lipschitz constant L_j, the claim implies that we get essentially the same Lipschitz constant L_j for ϕ_n^+ (with arbitrary $n \geq j$) at all those points at which ϕ_j^+ is stabilised already.

By this means, we are able to establish a decomposition of ϕ^+ into Lipschitz graphs whose Hausdorff dimension equals d (see Lemma 2.2.5). By the countable stability of the Hausdorff dimension (see Lemma 2.2.4), this yields that $D_H(\Phi^+) = d$. Part (iii) and (iv) of Theorem B are not so easy to illustrate on this qualitative level since we need some understanding of the local densities of those sets which are not hit by future peaks. Still, despite some refinement, the arguments are very much based on the above observations.

To formalise ideas, we introduce

$$\Omega_j^n = \mathbb{T}^d \setminus \bigcup_{k=j}^{\infty} \bigcup_{l=M_{k-1}}^{\min\{n, 2K_k M_k\}} \mathcal{I}_k + l\omega, \qquad \Omega_j = \bigcap_{n\in\mathbb{N}} \Omega_j^n = \mathbb{T}^d \setminus \bigcup_{k=j}^{\infty} \bigcup_{l=M_{k-1}}^{2K_k M_k} \mathcal{I}_k + l\omega,$$

where $j, n \in \mathbb{N}$. A way to interpret these definitions in terms of our qualitative discussion is the following: by the recursive definition of $\mathcal{I}_j$ (cf. Section 4.2.1), the size of the M_{j-1}-th peak is about $|\mathcal{I}_j|$. Hence, Ω_j only contains points which are not hit by any peak that appears after M_{j-1} iterations. Likewise, Ω_j^n contains points at which ϕ_n^+ might stabilise in finite time, but at which new peaks could still appear at future iterations.

Observe that for $k \in \mathbb{N}$ we have $K_k M_k \leq K_0 \kappa^k \cdot 2K_{k-1}M_{k-1} \leq \ldots \leq K_0^{k+1}\kappa^{\sum_{\ell=1}^{k}\ell} 2^k M_0$ while $|\mathcal{I}_k| \leq \varepsilon_k = \tilde{\mathscr{C}}\tilde{\alpha}^{-M_{k-1}} \leq \tilde{\mathscr{C}}\tilde{\alpha}^{-K_0^{k-1}\kappa^{\sum_{\ell=1}^{k-2}\ell}2^{k-1}M_0}$ (cf. Proposition 4.2.17). Thus, we have

$2K_kM_k\varepsilon_k^d < \varepsilon_k^{d/2}$ for large enough k and hence,

$$\mathrm{Leb}_{\mathbb{T}^d}\left(\bigcup_{k=j}^{\infty}\bigcup_{l=M_{k-1}}^{2K_kM_k}\mathcal{I}_k + l\omega\right) < \sum_{k=j}^{\infty} V_d 2K_kM_k\varepsilon_k^d < \sum_{k=j}^{\infty} V_d\varepsilon_k^{d/2}, \tag{5.1.1}$$

for large enough j, where V_d is the volume of the d-dimensional unit ball. Thus, $\mathrm{Leb}_{\mathbb{T}^d}(\Omega_j) > 0$ for large enough $j \in \mathbb{N}$.

There might still be points which get hit by infinitely many peaks so that no eventual stabilisation occurs. These are collected within

$$\Omega_\infty = \mathbb{T}^d \setminus \bigcup_{j\in\mathbb{N}}\Omega_j = \bigcap_{i=1}^{\infty}\bigcup_{k=i}^{\infty}\bigcup_{l=M_{k-1}}^{2K_kM_k}\mathcal{I}_k + l\omega.$$

In the following, we only consider the upper bounding lines ϕ_n^+ and the upper boundary graph ϕ^+. All of the results and proofs which are only stated in terms of ϕ^+ and ϕ_n^+ hold analogously for the lower bounding lines ϕ_n^- and the lower boundary graph ϕ^- as can be seen by considering f^{-1} instead of f.

The next proposition is the basis of our geometrical investigation of ϕ^+. Its proof, which is the technical core of this chapter, is given in Section 5.4. However, the statement should seem plausible to the reader in the light of the above discussion.

Proposition 5.1.2. *Let $f \in \mathcal{V}$. There are $\lambda > 0$ and $C > 0$ such that the following is true for sufficiently large j.*

(i) Suppose $\theta \in \Omega_j^n$ and $n > 2K_{j-1}M_{j-1} - M_{j-1} - 1$. Then $|\phi_n^+(\theta) - \phi_{n-1}^+(\theta)| \leq |c^+ - e^-| \cdot \alpha^{-\lambda(n-1)}$.

(ii) Suppose $\theta, \theta' \in \Omega_j^n$ and $n \in \mathbb{N}$. Then $\left|\phi_n^+(\theta) - \phi_n^+(\theta')\right| \leq L_j d(\theta,\theta')$ for some $L_j \leq \varepsilon_j^{-CK_{j-1}}$ independent of n.

5.2. Hausdorff, pointwise and information dimension

The information on the geometry of the curves ϕ_n^+ of the previous paragraph allows to determine the Hausdorff dimension of Φ^+ rather easily. The next statement and its proof are as in [GJ13, Proposition 5.1, Theorem 5.2 & Theorem 5.5].

Theorem 5.2.1. *Suppose $f \in \mathcal{V}$. Then the following statements hold:*

(i) $D_H(\Phi^+) = d$,

(ii) μ_{ϕ^+} is d-rectifiable and exact dimensional with $d_{\mu_{\phi^+}} = d$.

Proof. For each $j \in \mathbb{N} \cup \{\infty\}$ set $\Phi^+\restriction_{\Omega_j} = \{(\theta, \phi^+(\theta)) \colon \theta \in \Omega_j\}$. First, we want to show that $\Phi^+\restriction_{\Omega_j}$ is the image of a bi-Lipschitz continuous function g_j for all $j \in \mathbb{N}$. Define $g_j : \Omega_j \ni \theta \mapsto (\theta, \phi^+(\theta)) \in \Omega_j \times X$ for all $j \in \mathbb{N} \cup \{\infty\}$. We have that $g_j(\Omega_j) = \Phi^+\restriction_{\Omega_j}$ and $d_{\mathbb{T}^d \times X}(g_j(\theta), g_j(\theta')) \geq d(\theta, \theta')$ for all $\theta, \theta' \in \Omega_j$. We may assume without loss of generality that j is large enough[1] so that Proposition 5.1.2 (ii) yields that $\phi_n^+\restriction_{\Omega_j}$ is Lipschitz continuous with Lipschitz constant L_j independent of n. Since $\phi^+\restriction_{\Omega_j} = \lim_{n\to\infty} \phi_n^+\restriction_{\Omega_j}$, we also get that $\phi^+\restriction_{\Omega_j}$ is Lipschitz continuous with the same constant and therefore

$$d_{\mathbb{T}^d \times X}(g_j(\theta), g_j(\theta')) \leq \left(1 + L_j\right) d(\theta, \theta'),$$

for all $\theta, \theta' \in \Omega_j$ and $j \in \mathbb{N}$. Hence, g_j is bi-Lipschitz continuous for each $j \in \mathbb{N}$.

(i) We want to make use of the fact that the Hausdorff dimension is countably stable (see Lemma 2.2.4). Because of the bi-Lipschitz continuity of g_j and Lemma 2.2.5, we get that $D_H\left(\Phi^+\restriction_{\Omega_j}\right) = D_H(\Omega_j)$. Since $\mathrm{Leb}_{\mathbb{T}^d}(\Omega_j) > 0$ for large enough j, this implies $D_H\left(\Phi^+\restriction_{\Omega_j}\right) = d$ (cf. Lemma 2.2.2). What is left to show is that $D_H\left(\Phi^+\restriction_{\Omega_\infty}\right) \leq d$. Observe that Ω_∞ is a lim sup set. With a proper relabelling and doing a similar estimation as in (5.1.1), we can use Lemma 2.2.3 to conclude that $D_H(\Omega_\infty) \leq s$ for all $s > 0$. Therefore, $D_H(\Omega_\infty) = 0$. Further, $\Phi^+\restriction_{\Omega_\infty} \subset \Omega_\infty \times [e^-, c^+]$ and hence $D_H\left(\Phi^+\restriction_{\Omega_\infty}\right) \leq D_H(\Omega_\infty) + D_B([e^-, c^+]) = 1 \leq d$, applying Theorem 2.2.7.

(ii) Note that by definition, μ_{ϕ^+} is absolutely continuous with respect to $\mathcal{H}^d\restriction_{\Phi^+}$. We have that $\mu_{\phi^+}\left(\Phi^+\restriction_{\Omega_\infty}\right) = 0$ and therefore μ_{ϕ^+} is also absolutely continuous with respect to $\mathcal{H}^d\restriction_{\Phi^+ \setminus \Phi^+\restriction_{\Omega_\infty}}$. Since $\Phi^+ \setminus \Phi^+\restriction_{\Omega_\infty} = \bigcup_{j\in\mathbb{N}} \Phi^+\restriction_{\Omega_j}$ is a countably d-rectifiable set–using the observation from the beginning of the proof–we get that μ_{ϕ^+} is d-rectifiable, too. Now, by applying Corollary 2.2.12, we obtain that μ_{ϕ^+} is exact dimensional with pointwise dimension $d_{\mu_{\phi^+}} = d$. □

Remark. By the remark in Section 2.2.2, we immediately get that the information dimension of μ_{ϕ^+} equals d.

5.3. Minimality and box-counting dimension

For $n \in \mathbb{N}_0$, we denote by $\tilde{\mathcal{I}}_n$ the $\varepsilon_n/2$-neighbourhood of $\mathcal{I}_n$, that is, $\tilde{\mathcal{I}}_n = \bigcup_{\theta\in\mathcal{I}_n} B_{\varepsilon_n/2}(\theta)$. Set

$$\tilde{\Omega}_\infty = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} \bigcup_{l=M_{k-1}}^{2K_k M_k} \tilde{\mathcal{I}}_k + l\omega.$$

[1] Observe that for $j \leq J$, we have $\Phi^+\restriction_{\Omega_j} \subseteq \Phi^+\restriction_{\Omega_J}$ because $\Omega_j \subseteq \Omega_J$.

Lemma 5.3.1. *Suppose $\theta \notin \tilde{\Omega}_\infty$. Then there exists $j_0 \in \mathbb{N}$ such that for all integers $j \geq j_0$ we have $\theta \in \Omega_j$ and*

$$\mathrm{Leb}_{\mathbb{T}^d}(B_{\varepsilon_n/2}(\theta) \cap \Omega_j)/\,\mathrm{Leb}_{\mathbb{T}^d}(B_{\varepsilon_n/2}(\theta)) \to 1, \tag{5.3.1}$$

for $n \to \infty$.

Proof. By the assumptions, there is $j_0 \in \mathbb{N}$ such that $\theta \notin \bigcup_{k=j_0}^{\infty} \bigcup_{l=M_{k-1}}^{2K_kM_k} \tilde{\mathcal{I}}_k + l\omega$. Fix an arbitrary $j \geq j_0$ and observe that

$$B_{\varepsilon_n/2}(\theta) \cap \left(\bigcup_{k=j}^{n} \bigcup_{l=M_{k-1}}^{2K_kM_k} \mathcal{I}_k + l\omega \right) = \emptyset$$

for $n \geq j$ by definition of $\tilde{\mathcal{I}}_k$. Thus,

$$B_{\varepsilon_n/2}(\theta) \cap \Omega_j = B_{\varepsilon_n/2}(\theta) \setminus \bigcup_{k=j}^{\infty} \bigcup_{l=M_{k-1}}^{2K_kM_k} \mathcal{I}_k + l\omega = B_{\varepsilon_n/2}(\theta) \setminus \bigcup_{k=n+1}^{\infty} \bigcup_{l=M_{k-1}}^{2K_kM_k} \mathcal{I}_k + l\omega.$$

Similarly as in (5.1.1), we get $\mathrm{Leb}_{\mathbb{T}^d}\left(\bigcup_{k=n+1}^{\infty} \bigcup_{l=M_{k-1}}^{2K_kM_k} \mathcal{I}_k + l\omega\right) < \sum_{k=n+1}^{\infty} V_d \varepsilon_k^{d/2}$ for large enough n, where V_d normalises the Lebesgue measure. □

Lemma 5.3.2. *Suppose $\theta \in \tilde{\Omega}_\infty$. For each $\ell \in \mathbb{N}$, there are arbitrarily large j such that*

$$B_{\varepsilon_j/2}(\theta) \subseteq \Omega_{j+1}^{2K_{j+\ell}M_{j+\ell}} \tag{5.3.2}$$

and

$$\mathrm{Leb}_{\mathbb{T}^d}\left(B_{\varepsilon_j/2}(\theta)\right) - \mathrm{Leb}_{\mathbb{T}^d}\left(B_{\varepsilon_j/2}(\theta) \cap \Omega_{j+1}\right) < \varepsilon_{j+\ell}. \tag{5.3.3}$$

Proof. For $n \in \mathbb{N}$, we define

$$j_n = \max\left\{p \in \mathbb{N}_0 \colon \exists l \in \left[M_{p-1}, \min\left\{n, 2K_pM_p\right\}\right] \text{ such that } \theta \in \tilde{\mathcal{I}}_p + l\omega\right\}$$

and let $l_n \in \left[M_{j_n-1}, 2K_{j_n}M_{j_n}\right]$ be the corresponding time such that $\theta \in \tilde{\mathcal{I}}_{j_n} + l_n\omega$, where uniqueness is guaranteed by $(\mathcal{F}1)'_{j_n}$. Note that j_n and l_n are well-defined for sufficiently large n and $j_n \overset{n\to\infty}{\longrightarrow} \infty$ because $\theta \in \tilde{\Omega}_\infty$.

Let $\theta_* \in \bigcap_{n=0}^{\infty} \mathcal{I}_n$. Note that $d(\theta_* + l\omega, \theta) < \frac{3}{2}\varepsilon_{j_n}$ for all l for which $\theta \in \tilde{\mathcal{I}}_{j_n} + l\omega$. Now, suppose there is $k \in \mathbb{N}$ such that $\theta \in \tilde{\mathcal{I}}_{j_n} + l_n\omega + k\omega$. Then

$$d\,(k\omega, 0) = d\,(\theta_* + (l_n + k)\omega, \theta_* + l_n\omega) \leq d\,(\theta_* + (l_n + k)\omega, \theta) + d\,(\theta, \theta_* + l_n\omega) < 3\varepsilon_{j_n}.$$

As ω is Diophantine, this means $\mathscr{C}|k|^{-\eta} < d(k\omega, 0) < 3\varepsilon_{j_n}$ and hence

$$|k| > \tilde{c}\varepsilon_{j_n}^{-1/\eta}, \tag{5.3.4}$$

for some $\tilde{c} > 0$. Define

$$J_n = \max\left\{P\colon 2K_P M_P < \tilde{c}\varepsilon_{j_n}^{-1/\eta}\right\}.$$

By (5.3.4), we have

$$B_{\varepsilon_{j_n}/2}(\theta) \subseteq \Omega_{j_n+1}^{2K_{J_n}M_{J_n}}.$$

Since $j_n/J_n \xrightarrow{n\to\infty} 0$, we have thus shown that for any $\ell \in \mathbb{N}$ there is arbitrarily large j such that $B_{\varepsilon_j/2}(\theta) \subseteq \Omega_{j+1}^{2K_{j+\ell}M_{j+\ell}}$.

Given $\ell \in \mathbb{N}$, assume j is such that (5.3.2) holds. Then,

$$B_{\varepsilon_j/2}(\theta) \cap \Omega_{j+1} = B_{\varepsilon_j/2}(\theta) \setminus \bigcup_{k=j+1}^{\infty} \bigcup_{l=M_{k-1}}^{2K_kM_k} \mathcal{I}_k + l\omega = B_{\varepsilon_j/2}(\theta) \setminus \bigcup_{k=j+\ell+1}^{\infty} \bigcup_{l=M_{k-1}}^{2K_kM_k} \mathcal{I}_k + l\omega.$$

Finally, $\mathrm{Leb}_{\mathbb{T}^d}\left(\bigcup_{k=j+\ell+1}^{\infty}\bigcup_{l=M_{k-1}}^{2K_kM_k} \mathcal{I}_k + l\omega\right) < \sum_{k=j+\ell+1}^{\infty} V_d\varepsilon_k^{d/2} < \varepsilon_{j+\ell}$ for large enough j. □

Corollary 5.3.3. *Let $f \in \mathcal{V}$. If $\phi = \phi^+$ a.s. and ϕ is an upper semi-continuous invariant graph, then $\phi = \phi^+$. In other words, ϕ^+ is the unique upper semi-continuous invariant graph in its equivalence class. Further,*

$$\phi^+\left(\overline{B_r(\theta)}\right) \subseteq \overline{\phi^+\left(B_r(\theta)\right)}, \tag{5.3.5}$$

for all $\theta \in \mathbb{T}^d$ and all $r > 0$.

Proof. We first show (5.3.5). Let $\theta \in \mathbb{T}^d$ and $r > 0$ be given and let $\theta_0 \in \partial B_r(\theta) = \overline{B_r(\theta)} \setminus B_r(\theta)$.

First, consider the case where $\theta_0 \notin \tilde{\Omega}_\infty$ and let j be as in Lemma 5.3.1. Equation (5.3.1) yields that for every $\rho > 0$ there is $\theta' \in B_r(\theta)\cap B_\rho(\theta_0)$ such that $\theta' \in \Omega_j$. Without loss of generality we may assume that j is large enough so that Proposition 5.1.2(ii) gives

$$\left|\phi_n^+(\theta_0) - \phi_n^+(\theta')\right| \leq L_j d(\theta_0, \theta')$$

for arbitrary n and thus $|\phi^+(\theta_0) - \phi^+(\theta')| \leq L_j d(\theta_0,\theta') \leq L_j\rho$ as $\phi_n^+ \to \phi^+$ point-wise. Sending ρ to zero proves $\phi^+(\theta_0) \in \overline{\phi^+(B_r(\theta))}$.

Now, suppose $\theta_0 \in \tilde{\Omega}_\infty$ and let $\delta > 0$. Lemma 5.3.2 yields that there is arbitrarily large $j \in \mathbb{N}$ such that $\theta_0 \in \Omega_j^{2K_{j+2}M_{j+2}}$. For sufficiently large j, equation (5.3.3) gives $B_r(\theta) \cap B_{\delta\varepsilon_j^{CK_{j-1}}}(\theta_0) \cap \Omega_j \neq \emptyset$, where we may choose C such that $L_j \leq \varepsilon_j^{-CK_{j-1}}$ (see Proposition 5.1.2 (ii)). Let $\theta' \in B_r(\theta)\cap B_{\delta\varepsilon_j^{CK_{j-1}}}(\theta_0)\cap\Omega_j$. Then $\left|\phi_{2K_jM_j}^+(\theta_0) - \phi_{2K_jM_j}^+(\theta')\right| \leq \delta$ by Proposition 5.1.2 (ii). Without loss of generality we may further assume that j is

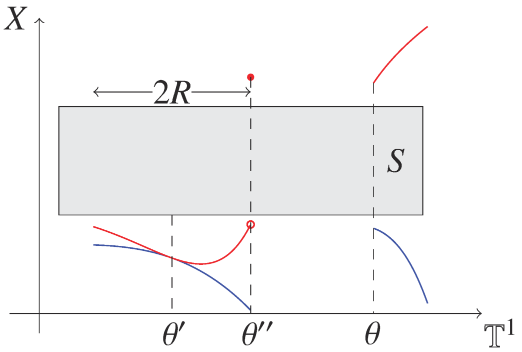

Figure 5.2. The 1-dimensional case: Assuming a gap within the minimal set implies the existence of a point $(\theta'', \phi^+(\theta''))$ which is isolated from one side (here, from the left). This contradicts Corollary 5.3.3.

large enough to ensure $\left|\phi^+(\theta_0) - \phi^+_{2K_jM_j}(\theta_0)\right| \leq \delta$ and $|c^+ - e^-| \cdot \sum_{k=2K_jM_j}^{\infty} \alpha^{-\lambda k} \leq \delta$, for λ as in Proposition 5.1.2(i). This eventually gives

$$
\begin{aligned}
&\left|\phi^+(\theta_0) - \phi^+(\theta')\right| \\
&\leq \left|\phi^+(\theta_0) - \phi^+_{2K_jM_j}(\theta_0)\right| + \left|\phi^+_{2K_jM_j}(\theta_0) - \phi^+_{2K_jM_j}(\theta')\right| + \left|\phi^+_{2K_jM_j}(\theta') - \phi^+(\theta')\right| \\
&\leq 3\delta,
\end{aligned}
$$

where we used Proposition 5.1.2(i) (again, assuming large enough j) to estimate the last term.

Given arbitrary $\theta \in \mathbb{T}^d$ and $r > 0$, we have thus shown that for each $\theta_0 \in \partial B_r(\theta)$ there is a sequence $\theta_n \overset{n\to\infty}{\longrightarrow} \theta_0$ within $B_r(\theta)$ such that $\phi^+(\theta_0) = \lim_{n\to\infty} \phi^+(\theta_n)$. Hence, (5.3.5) holds. In fact, the construction shows that even if $\phi = \phi^+$ only *almost* surely, we still find a sequence $\tilde{\theta}_n \overset{n\to\infty}{\longrightarrow} \theta_0$ within $B_r(\theta)$ such that $\phi(\tilde{\theta}_n) = \phi^+(\tilde{\theta}_n) \overset{n\to\infty}{\longrightarrow} \phi^+(\theta_0)$. Thus, if ϕ is upper semi-continuous, this necessarily yields $\phi \geq \phi^+$. On the other hand, if ϕ is invariant, its graph is contained entirely within the maximal invariant set Λ so that $\phi \leq \phi^+$. Thus, $\phi = \phi^+$. □

For the proof of the next statement, it is important to note that there is no lower semi-continuous invariant graph that coincides almost surely with ϕ^+. This can be seen as follows. From Corollary 2.1.15, we get that there is no lsc invariant graph *above* ϕ^+ coinciding with ϕ^+ almost surely. On the other hand, given a lsc invariant graph ϕ equivalent to and below ϕ^+, note that there can be no θ' with $\phi(\theta') < \phi^-(\theta')$ as this would imply the existence of a dense set in $\mathbb{T}^d$ on which $\phi \leq \gamma^-$–contradicting the assumed lower semi-continuity and $\phi = \phi^+ > \phi^- \geq \gamma^-$ a.s. Hence, a lsc invariant graph ϕ equivalent to ϕ^+ would have to be contained in $[\phi^-, \phi^+]$. Consequently, ϕ would have to coincide with ϕ^+ at the pinched points. Observe that the sink-source orbit constructed in Chapter 4 is contained in the set of pinched points. Hence, the closed and invariant set $[\phi, \phi^+]$ (cf. Corollary 2.1.13) contains a sink-source orbit and thus, by Corollary 2.1.28, an invariant graph $\tilde{\phi}$ with positive Lyapunov exponent. This contradicts the equivalence of $\tilde{\phi}$ and ϕ^+ (see also [Jäg03, Lemma 3.2]). Likewise, we see that there is no usc invariant graph equivalent to ϕ^-.

Theorem 5.3.4. *Let $f \in \mathcal{V}$. Then $[\phi^-, \phi^+]$ is minimal. As a consequence, $D_B(\Phi^-) = D_B(\Phi^+) = d + 1$.*

Proof. As ϕ^- and ϕ^+ are lower and upper semi-continuous invariant graphs, respectively, $[\phi^-, \phi^+]$ is a compact invariant set.

For a contradiction, assume $[\phi^-, \phi^+]$ is not minimal. Then there is a proper subset $M \subset [\phi^-, \phi^+]$ which is compact and invariant. Denote the respective upper and lower boundary graphs by ϕ^+_M and ϕ^-_M. Theorem 2.1.23 and Corollary 5.3.3 as well as Lemma 2.1.10 and the above discussion yield that $\phi^\pm_M = \phi^\pm$. Hence, there have to be $\theta \in \mathbb{T}^d$ and $x \in \left(\phi^-_M(\theta), \phi^+_M(\theta)\right)$ with $(\theta, x) \notin M$. Since M is compact, there actually is an open strip $S = B_{\varepsilon_1}(\theta_0) \times B_{\varepsilon_2}(x_0)$ with $\varepsilon_1, \varepsilon_2 > 0$ centred at some $(\theta_0, x_0) \in \mathbb{T}^d \times X$ such that $(\theta, x) \in S$ and $S \cap M = \emptyset$.

By Lemma 2.1.17 (a), we may assume without loss of generality that there is a pinched point $\theta' \in B_{\varepsilon_1}(\theta_0)$ with $\phi^-(\theta') = \phi^+(\theta') < x_0 - \varepsilon_2$. In other words, Φ^- and Φ^+ have a common point below S. By continuity of ϕ^+ at the pinched points (see Lemma 2.1.17 (b)), we have that $\Phi^+\upharpoonright_{B_r(\theta')} = \Phi^+ \bigcap B_r(\theta') \times X$ is below S for small enough $r > 0$. Denote by R the supremum of all such r and assume without loss of generality that $\overline{B_R(\theta')} \subseteq B_{\varepsilon_1}(\theta_0)$. Then, $\Phi^+\upharpoonright_{B_R(\theta')}$ is below S, while $\Phi^+\upharpoonright_{B_{R+\delta}(\theta')}$ necessarily contains points above S for each $\delta > 0$. Hence, there is $\theta'' \in \partial B_R(\theta')$ such that $(\theta'', \phi^+(\theta''))$ is above S, contradicting Corollary 5.3.3 (cf. Figure 5.2). This proves the desired minimality.

As an immediate consequence, we have $\overline{\Phi^-} = \overline{\Phi^+} = [\phi^-, \phi^+]$ and so, by the remark in Section 2.2.1, $D_B(\Phi^-) = D_B(\Phi^+) = D_B([\phi^-, \phi^+])$. Since $\phi^- < \phi^+$ a.s., we further have $D_B([\phi^-, \phi^+]) = d + 1$. □

5.4. Convergence and Lipschitz continuity of iterated bounding lines

We now turn to the proof of Proposition 5.1.2. It is based on the bounds on the derivatives in $(\mathcal{A}1)$–$(\mathcal{A}3)$ and $(\mathcal{A}10)$ as well as $(\mathcal{A}5)$ and the combinatorial observations from Section 4.2.1[2].

As before, we consider the iterated upper bounding lines only. Given fixed $n \in \mathbb{N}$ and $\theta \in \mathbb{T}^d$, we set

$$\theta^k = \theta - (n-k)\omega \quad \text{and} \quad x^k = f^k_{\theta^0}(c^+)$$

such that $\phi^+_k(\theta^k) = x^k$. Note the difference to $\theta_k = \theta + k\omega$ and $x_k = f^k_\theta(x)$ as introduced in Section 4.2.1.

[2] Notice that by definition of $\mathcal{V}$ and Proposition 4.2.17, the n-th critical region is non-empty and $|\mathcal{I}_n| \leq \varepsilon_n$ while $(\mathcal{F}1)'_n$ and $(\mathcal{F}2)_n$ are verified for arbitrary n.

Let $p \in \mathbb{N}$ and consider a finite orbit $\{(\theta^0, x), \dots, f^n(\theta^0, x)\}$ which initially verifies $(\mathcal{B}1)_p$ and hits $\mathcal{I}_p$ only at $\theta^0 + n\omega$. Lemma 4.2.6 provides us with a lower bound on the times spent in the contracting region between any time k and only such following times at which the orbit hits $\mathcal{I}_{p-1}$. If we want a lower bound on the times in the contracting region between any two consecutive moments $k < l$, we have to deal with the fact that Lemma 4.2.3 might allow the orbit to stay in the expanding region for $M_{p-1} + 1$ times after hitting $\mathcal{I}_{p-1}$. This is taken care of in the following corollary of Lemma 4.2.3 and Lemma 4.2.6.

For $\theta \in \mathbb{T}^d$ and $0 \le k \le n$, set

$$p_k^n(\theta) = \max\left\{p \in \mathbb{N}_0 \colon \exists l \in [M_{p-1}, \min\{n, n-k+M_p+1\}] \text{ with } \theta_{-l} = \theta - l\omega \in \mathcal{I}_p\right\}$$

with $\max \emptyset = -1$. The reader may also find the following–and obviously equivalent–characterisation of $p_k^n(\theta)$ useful

$$p_k^n(\theta) = \max\left\{p \in \mathbb{N}_0 \colon \exists l \in [\max\{0, k - M_p - 1\}, n - M_{p-1}] \text{ with } \theta^l = \theta - (n-l)\omega \in \mathcal{I}_p\right\}.$$

Observe that $p_\ell^n(\theta)$ and $p_{k-\ell}^{n-\ell}(\theta)$ are non-increasing in ℓ.

Corollary 5.4.1. *Let $f \in \mathcal{V}$ and suppose $(\theta^0, x) = (\theta - n\omega, x)$ satisfies $(\mathcal{B}1)_{p_0^n(\theta)+1}$. Then*

$$\mathcal{P}_k^n(\theta^0, x) \ge b_{p_k^n(\theta)+1}\left(n - k - \sum_{j=0}^{p_k^n(\theta)} (M_j + 2)\right) \quad \textit{for each } k = 0, \dots, n-1. \tag{5.4.1}$$

Proof. For integers $p \ge -1$, set

$$\Theta_p = \left\{(\theta, x, n) \in \mathbb{T}^d \times C \times \mathbb{N} \colon p_0^n(\theta) \le p \text{ and } (\theta - n\omega, x) \text{ satisfy } (\mathcal{B}1)_{p_0^n(\theta)+1}\right\}.$$

We say (5.4.1) *holds within* Θ_p if (5.4.1) is true for all $(\theta, x, n) \in \Theta_p$. We show by induction on p that (5.4.1) holds within Θ_p for all p. Note that within Θ_{-1}, 5.4.1 follows directly from $(\mathcal{A}5)$.

Suppose there is an integer $p_0 \ge -1$ so that (5.4.1) holds within Θ_{p_0}. Set $p = p_0 + 1$ and fix $(\theta, x, n) \in \Theta_p \setminus \Theta_{p_0}$ which is assumed to be non-empty without loss of generality. Let $\mathcal{L}$ be the largest positive integer not bigger than $n - M_{p-1}$ such that $\theta^{\mathcal{L}} \in \mathcal{I}_p$ and assume without loss of generality that $\mathcal{L} < n$. Note that $p_{\mathcal{L}}^n(\theta) = p$. First, let $k \in [\mathcal{L}, n-1]$. There are two cases to be considered.

(a) Suppose $\mathcal{L} \ge n - M_p - 2$. Then $\mathcal{L} \in [\max\{0, k - M_p - 1\}, n - M_{p-1}]$ for all $k \le n-1$, by definition of $\mathcal{L}$. Hence, $p_k^n(\theta) = p$ for all $k \in [\mathcal{L}, n-1]$ since $\theta^{\mathcal{L}} \in \mathcal{I}_p$. Thus, $k \ge \mathcal{L} \ge n - M_{p_k^n(\theta)} - 2$ and so $M_{p_k^n(\theta)} \ge n - k - 2$ so that (5.4.1) holds trivially.

(b) Suppose $\mathcal{L} < n - M_p - 2$. First, consider $k \ge \mathcal{L} + M_p + 2$. Then $p_k^n(\theta) < p$ and hence $p_{k-(\mathcal{L}+M_p+2)}^{n-(\mathcal{L}+M_p+2)}(\theta) \le p_k^n(\theta) < p$. Further by Lemma 4.2.3, $f^{\mathcal{L}+M_p+2}(\theta^0, x)$ satisfies

$(\mathcal{B}1)_{p+1}$, and thus certainly $(\mathcal{B}1)_{p_0+1}$. Hence, we get

$$\begin{aligned}
\mathcal{P}_k^n(\theta^0, x) &= \mathcal{P}_{k-(\mathcal{L}+M_p+2)}^{n-(\mathcal{L}+M_p+2)}(f^{\mathcal{L}+M_p+2}(\theta^0, x)) \\
&\geq b_{p_{k-(\mathcal{L}+M_p+2)}^{n-(\mathcal{L}+M_p+2)}(\theta)+1}\left(n-k-\sum_{j=0}^{p_{k-(\mathcal{L}+M_p+2)}^{n-(\mathcal{L}+M_p+2)}(\theta)}(M_j+2)\right) \\
&\geq b_{p_k^n(\theta)+1}\left(n-k-\sum_{j=0}^{p_k^n(\theta)}(M_j+2)\right),
\end{aligned}$$

where the first estimate follows by the induction hypothesis and the last estimate from the fact that b_q is decreasing in q. Now, if $k \in [\mathcal{L}, \mathcal{L}+M_p+1]$, then

$$\begin{aligned}
\mathcal{P}_k^n(\theta^0, x) &= \mathcal{P}_k^{\mathcal{L}+M_p+2}(\theta^0, x) + \mathcal{P}_{\mathcal{L}+M_p+2}^n(\theta^0, x) \geq \mathcal{P}_{\mathcal{L}+M_p+2}^n(\theta^0, x) \\
&\geq b_{p_{\mathcal{L}+M_p+2}^n(\theta)+1}\left(n-\mathcal{L}-M_p-2-\sum_{j=0}^{p_{\mathcal{L}+M_p+2}^n(\theta)}(M_j+2)\right) \\
&\geq b_{p_k^n(\theta)+1}\left(n-k-M_p-2-\sum_{j=0}^{p_{\mathcal{L}+M_p+2}^n(\theta)}(M_j+2)\right) \\
&\geq b_{p_k^n(\theta)+1}\left(n-k-\sum_{j=0}^{p_k^n(\theta)}(M_j+2)\right),
\end{aligned}$$

where the last estimate holds since $p_k^n(\theta) = p$ for $k \leq \mathcal{L}+M_p+1$.

We have thus shown

$$\mathcal{P}_k^n(\theta^0, x) \geq b_{p_k^n(\theta)+1}\left(n-k-\sum_{j=0}^{p_k^n(\theta)}(M_j+2)\right) \tag{5.4.2}$$

for $k \in [\mathcal{L}, n-1]$.

It remains to consider $k < \mathcal{L}$. Since $p_k^n(\theta) \geq p_{\mathcal{L}}^n(\theta) = p$, we obtain

$$\begin{aligned}
\mathcal{P}_k^n(\theta^0, x) = \mathcal{P}_k^{\mathcal{L}}(\theta^0, x) + \mathcal{P}_{\mathcal{L}}^n(\theta^0, x) &\geq b_{p+1}(\mathcal{L}-k) + b_{p_{\mathcal{L}}^n(\theta)+1}\left(n-\mathcal{L}-\sum_{j=0}^{p_{\mathcal{L}}^n(\theta)}M_j+2\right) \\
&\geq b_{p+1}\left(n-k-\sum_{j=0}^{p}M_j+2\right),
\end{aligned}$$

where we used Lemma 4.2.6 and equation (5.4.2) in the first estimate. As (θ, x, n) was arbitrary in $\Theta_p \setminus \Theta_{p_0}$, this shows that (5.4.1) holds within Θ_p. □

For $k, n \in \mathbb{N}$, set $i_k^n = \max\{l \colon n-k \geq 2K_l M_l - M_l - 1\}$.

Proposition 5.4.2. *Suppose $\theta \in \Omega_j^n$ for some $j \in \mathbb{N}$. Then $i_k^n \geq p_k^n(\theta)$ for all $0 \leq k \leq n-(2K_{j-1}M_{j-1}-M_{j-1}-1)$.*

Proof. Note that by the assumptions $i_k^n \geq j-1$. Thus, without loss of generality we may assume $p_k^n(\theta) > j-1$. By definition of $p_k^n(\theta)$, there is $l \in [M_{p_k^n(\theta)-1}, n-k+M_{p_k^n(\theta)}+1]$ such that $\theta - l\omega \in \mathcal{I}_{p_k^n(\theta)}$. Since $\theta \in \Omega_j^n$, this implies $l > 2K_{p_k^n(\theta)}M_{p_k^n(\theta)}$ and thus, $n-k > 2K_{p_k^n(\theta)}M_{p_k^n(\theta)} - M_{p_k^n(\theta)} - 1$ which means $i_k^n \geq p_k^n(\theta)$. □

Proof of Proposition 5.1.2. Let $\theta \in \Omega_j^n$ and let $\mathcal{L}$ be the smallest positive integer such that $\theta^0 - \mathcal{L}\omega = \theta - (\mathcal{L}+n)\omega \in \mathcal{I}_{p_n^n(\theta)}$. Then $(\theta^0 - (\mathcal{L}-1)\omega, c^+)$ satisfies $(\mathcal{B}1)_{p_n^n(\theta)+1}$ because of $(\mathcal{F}1)_{p_n^n(\theta)}$. By the monotonicity of the fibre maps and by $(\mathcal{A}4)$, we have the implication

$$f^{\mathcal{L}-1+k}_{\theta^0-(\mathcal{L}-1)\omega}(c^+) \in C \implies f^k_{\theta^0}(c^+) \in C,$$

for all $k \geq 0$. Further, we observe that $p_0^n(\theta) = p_{\mathcal{L}-1}^{\mathcal{L}-1+n}(\theta)$ and actually $p_k^n(\theta) = p_{\mathcal{L}-1+k}^{\mathcal{L}-1+n}(\theta)$ for all $k = 0, \ldots, n$. By Corollary 5.4.1, we thus get

$$\begin{aligned}\mathcal{P}_k^n(\theta^0, c^+) &\geq \mathcal{P}^{\mathcal{L}-1+n}_{\mathcal{L}-1+k}(\theta^0 - (\mathcal{L}-1)\omega, c^+) \geq b_{p_k^n(\theta)+1}\left(n-k-\sum_{\ell=0}^{p_k^n(\theta)}(M_\ell+2)\right) \\ &\overset{\text{Proposition 5.4.2}}{\geq} b_{i_k^n+1}\left(n-k-\sum_{\ell=0}^{i_k^n}(M_\ell+2)\right),\end{aligned} \tag{5.4.3}$$

for $0 \leq k \leq n-(2K_{j-1}M_{j-1}-M_{j-1}-1)$. Now, note that $\sum_{\ell=0}^{i_k^n}(M_\ell+2) \leq 3/2M_{i_k^n}$ for large enough i_k^n (and hence, for large enough j since $i_k^n \geq j-1$). Further, $(n-k)/K_{i_k^n} \geq 2M_{i_k^n} - M_{i_k^n}/K_{i_k^n} - 1/K_{i_k^n}$ by definition of i_k^n. Thus for large enough j, we have $\sum_{\ell=0}^{i_k^n}(M_\ell+2) \leq (n-k)/K_{i_k^n}$ and so by (5.4.3)

$$\mathcal{P}_k^n(\theta^0, c^+) \geq b_{i_k^n+1}(1-1/K_{i_k^n})(n-k) \geq b^2(n-k). \tag{5.4.4}$$

By $(\mathcal{A}1)$–$(\mathcal{A}3)$, we hence have

$$\begin{aligned}&|\phi_n^+(\theta) - \phi_{n-1}^+(\theta)| \\ &= \phi_{n-1}^+(\theta) - \phi_n^+(\theta) = \left(\phi_0^+(\theta^1) - \phi_1^+(\theta^1)\right) \cdot \prod_{k=1}^{n-1} \frac{\phi_k^+(\theta^{k+1}) - \phi_{k+1}^+(\theta^{k+1})}{\phi_{k-1}^+(\theta^k) - \phi_k^+(\theta^k)} \\ &\leq |c^+ - e^-| \cdot \prod_{k=1}^{n-1} \frac{f_{\theta^k}\left(\phi_{k-1}^+(\theta^k)\right) - f_{\theta^k}\left(\phi_k^+(\theta^k)\right)}{\phi_{k-1}^+(\theta^k) - \phi_k^+(\theta^k)} \leq |c^+ - e^-| \cdot \alpha^{p\left((n-1)-\mathcal{P}_1^n(\theta^0,c^+)\right)-2\mathcal{P}_1^n(\theta^0,c^+)/p} \\ &\overset{(5.4.4)}{\leq} |c^+ - e^-| \cdot \alpha^{\left(p(1-b^2)-2b^2/p\right)(n-1)},\end{aligned}$$

where we assumed–without loss of generality–that $\phi^+_{k-1}(\theta^k) - \phi^+_k(\theta^k) > 0$ for all $k \in \{1,\ldots,n\}$. This proves the first part with $\lambda = 2b^2/p - p(1-b^2) > 0$.

Let $\wp^n_k(\theta,\theta') = \#\left\{\ell \in [k, n-1] \cap \mathbb{N}_0 \colon x^\ell, x'^\ell \in C\right\}$ for $\theta, \theta' \in \mathbb{T}^d$ where $x^\ell = f^\ell_{\theta^0}(c^+)$ and $x'^\ell = f^\ell_{\theta'^0}(c^+)$. By induction on n, we first show that for all $n \in \mathbb{N}$

$$|\phi^+_n(\theta) - \phi^+_n(\theta')| \le S\, d(\theta,\theta') \sum_{k=1}^{n} \alpha^{p\left(n-k-\wp^n_k(\theta,\theta')\right) - 2\wp^n_k(\theta,\theta')/p}. \tag{5.4.5}$$

For $n = 1$, this is $(\mathscr{A}10)$. Suppose (5.4.5) holds for some $n \in \mathbb{N}$. Since $\wp^n_k(\theta-\omega, \theta'-\omega) + \wp^{n+1}_n(\theta,\theta') = \wp^{n+1}_k(\theta,\theta')$, this yields

$$\begin{aligned}\left|\phi^+_{n+1}(\theta) - \phi^+_{n+1}(\theta')\right| &= \left|f_{\theta-\omega}\left(\phi^+_n(\theta-\omega)\right) - f_{\theta'-\omega}\left(\phi^+_n(\theta'-\omega)\right)\right| \\ &\le \alpha^{p\left(1-\wp^{n+1}_n(\theta,\theta')\right) - 2\wp^{n+1}_n(\theta,\theta')/p} |\phi^+_n(\theta-\omega) - \phi^+_n(\theta'-\omega)| \\ &\quad + S\, d(\theta-\omega, \theta'-\omega) \\ &\le S\, d(\theta,\theta') \sum_{k=1}^{n+1} \alpha^{p\left(n+1-k-\wp^{n+1}_k(\theta,\theta')\right) - 2\wp^{n+1}_k(\theta,\theta')/p}\end{aligned}$$

where we used $(\mathscr{A}1)$, $(\mathscr{A}3)$, and $(\mathscr{A}10)$ in the first estimate and the induction hypothesis in the last step. Hence, equation (5.4.5) holds.

Now, consider sufficiently large j and suppose $\theta, \theta' \in \Omega^n_j$. Suppose $n > 2K_{j-1}M_{j-1} - M_{j-1} - 1$ and observe that equation (5.4.4) gives

$$\begin{aligned}\wp^n_k(\theta,\theta') &\ge n - k - (2(n-k) - \mathcal{P}^n_k(\theta) - \mathcal{P}^n_k(\theta')) \\ &\ge n - k - 2(1-b^2)(n-k) = (2b^2-1)(n-k)\end{aligned}$$

for all $k = 0, \ldots, n - (2K_{j-1}M_{j-1} - M_{j-1} - 1)$. Plugging this into (5.4.5) yields

$$\begin{aligned}&|\phi^+_n(\theta) - \phi^+_n(\theta')| \\ &\le S\, d(\theta,\theta') \left(\sum_{k=1}^{n-2K_{j-1}M_{j-1}-M_{j-1}-1} \alpha^{\left(2p\left(1-b^2\right)-2(2b^2-1)/p\right)(n-k)} + \sum_{k=n-2K_{j-1}M_{j-1}-M_{j-1}}^{n} \alpha^{p\left(n-k-\wp^n_k(\theta,\theta')\right) - 2\wp^n_k(\theta,\theta')/p} \right) \\ &\le L_j d(\theta,\theta'),\end{aligned}$$

where $L_j = S \cdot \left(\sum_{l=2K_{j-1}M_{j-1}-M_{j-1}-1}^{\infty} \alpha^{\left(2p\left(1-b^2\right)-2(2b^2-1)/p\right)l} + \sum_{l=0}^{2K_{j-1}M_{j-1}-M_{j-1}} \alpha^{pl} \right)$. It is immediate that $|\phi^+_n(\theta) - \phi^+_n(\theta')| \le L_j d(\theta,\theta')$ holds for $n \le 2K_{j-1}M_{j-1} - M_{j-1} - 1$, too. Finally, observe that there is $C > 0$ (independent of j) such that $L_j \le \varepsilon_j^{-CK_{j-1}}$ for large enough j. □

6. Existence of strange attractors in forced differential equations

With the proof of Theorem C, we come to the last chapter of this thesis. The strategy we pursue is to construct an example of a skew product flow family $\hat{\Xi}$, which can be reduced to a family of first return maps $\hat{\hat{\Xi}}$ (on a suitable Poincaré section) such that $\hat{\hat{\Xi}} \in \mathcal{U}_\omega(X)$ (with an appropriate ω) . The considered example, or class of examples, is a family of additively forced logistic differential equations with a forcing term that vanishes outside a small region (see Section 6.2).

Once we have shown that the family of return maps $\hat{\hat{\Xi}}$ lies in $\mathcal{U}_\omega(X)$, it is not hard to see that this implies the occurrence of a non-smooth saddle-node bifurcation for $\hat{\Xi}$ (see Section 6.1). The genericity of such families $\hat{\Xi}$ will follow from Theorem A.

6.1. Prefatory observations

In this section, we provide some further technical background which goes beyond the general discussion in Section 2.1.

From now on, we consider families of (local) flows $\hat{\Xi}$ of the form (3.3.1) corresponding to some C^2-families of non-autonomous vector fields. That is, given $\hat{F} \in \mathscr{P}(X)$ with a fixed non-degenerate interval X and given a Diophantine rotation vector $\rho \in \mathbb{R}^D$ (for some integer $D \geq 2$), we consider

$$\Xi_\beta\colon U \subseteq \mathbb{R} \times \mathbb{T}^D \times X \to \mathbb{T}^D \times X, \quad (t, \theta, x) \mapsto (t \cdot \rho + \theta, \xi_\beta(t, \theta, x)), \tag{6.1.1}$$

where ξ_β is the maximal solution of

$$\partial_t \xi_\beta(t, \theta, x) = F_\beta(\rho_t(\theta), \xi_\beta(t, \theta, x)) \tag{6.1.2}$$

with $\xi_\beta(0, \theta, x) = x$ for each $(\theta, x) \in \Theta \times X$ and $\beta \in [0, 1]$, and U is the domain of ξ_β.

It is well-known that since F_β is C^2, the corresponding flow Ξ_β is also C^2 for each β (see, e.g. [Har64, Chapter V, Corollary 4.1]). As $\mathcal{U}_\omega(X)$ is defined in terms of C^2-estimates of the fibre maps, we need expressions for the derivatives of $\xi_\beta(\cdot, \theta, x)$. This is what the next section is about.

The second paragraph introduces suitable Poincaré sections and the according return maps in order to reduce the continuous time problem to a problem in discrete time.

6.1.1. Derivatives of the flow

As already mentioned, it is a well-known fact that for $\hat{F} \in \mathscr{F}_\rho(X)$, the map $(\beta, t, \theta, x) \mapsto \xi_\beta(t, \theta, x)$ is C^2 so that the task here is to differentiate (6.1.2) and express the solutions of the resulting ode's (sometimes referred to as *variational equations*) in terms of the (unknown) solution ξ_β of (6.1.2). By differentiating (6.1.2) with respect to x and ϑ, we get

$$\partial_t \partial_x \xi_\beta(t, \theta, x) = \partial_x F_\beta\left(t\rho + \theta, \xi_\beta(t, \theta, x)\right) \cdot \partial_x \xi_\beta(t, \theta, x), \tag{6.1.3}$$

$$\partial_t \partial_\vartheta \xi_\beta(t, \theta, x) = \partial_\vartheta F_\beta\left(t\rho + \theta, \xi_\beta(t, \theta, x)\right) + \partial_x F_\beta\left(t\rho + \theta, \xi_\beta(t, \theta, x)\right) \cdot \partial_\vartheta \xi_\beta(t, \theta, x). \tag{6.1.4}$$

Further, note that since $\xi_\beta(0, \theta, x) = x$, we have $\partial_x \xi_\beta(0, \theta, x) = 1$ and $\partial_\vartheta \xi_\beta(0, \theta, x) = 0$. Hence,

$$\partial_x \xi_\beta(t, \theta, x) = \exp\left(\int_0^t \partial_x F_\beta\left(s\rho + \theta, \xi_\beta(s, \theta, x)\right) ds\right) \tag{6.1.5}$$

and

$$\begin{aligned} &\partial_\vartheta \xi_\beta(t, \theta, x) \\ &= \int_0^t \partial_\vartheta F_\beta\left(s\rho + \theta, \xi_\beta(s, \theta, x)\right) \exp\left(\int_s^t \partial_x F_\beta\left(\tau\rho + \theta, \xi_\beta(\tau, \theta, x)\right) d\tau\right) ds. \end{aligned} \tag{6.1.6}$$

The expression for $\partial_x \xi_\beta(t, \theta, x)$ immediately shows monotonicity of ξ_β in x. However, observe that this already follows from the uniqueness of the solutions of (6.1.2), of course. We can differentiate (6.1.5) to get

$$\begin{aligned} \partial_x^2 \xi_\beta(t, \theta, x) = &\exp\left(\int_0^t \partial_x F_\beta\left(s\rho + \theta, \xi_\beta(s, \theta, x)\right) ds\right) \\ &\cdot \int_0^t \partial_x^2 F_\beta\left(s\rho + \theta, \xi_\beta(s, \theta, x)\right) \cdot \partial_x \xi_\beta(s, \theta, x)\, ds \\ = &\partial_x \xi_\beta(t, \theta, x) \cdot \int_0^t \partial_x^2 F_\beta\left(s\rho + \theta, \xi_\beta(s, \theta, x)\right) \cdot \partial_x \xi_\beta(s, \theta, x)\, ds \end{aligned} \tag{6.1.7}$$

and similarly

$$\begin{aligned} &\partial_\vartheta \partial_x \xi_\beta(t, \theta, x) \\ &= \partial_x \xi_\beta(t, \theta, x) \cdot \int_0^t \partial_\vartheta \partial_x F_\beta\left(s\rho + \theta, \xi_\beta(s, \theta, x)\right) + \partial_x^2 F_\beta\left(s\rho + \theta, \xi_\beta(s, \theta, x)\right) \cdot \partial_\vartheta \xi_\beta(s, \theta, x)\, ds. \end{aligned} \tag{6.1.8}$$

For simplicity, instead of further differentiating (6.1.6) with respect to ϑ, we differentiate (6.1.4) and solve the resulting problem with initial condition $\partial_\vartheta^2 \xi_\beta(0, \theta, x) = 0$ in order to obtain an expression for $\partial_\vartheta^2 \xi_\beta(t, \theta, x)$. Now,

$$\begin{aligned} \partial_t \partial_\vartheta^2 \xi_\beta(t, \theta, x) = &\partial_\vartheta^2 F_\beta\left(t\rho + \theta, \xi_\beta(t, \theta, x)\right) + 2\partial_\vartheta \partial_x F_\beta\left(t\rho + \theta, \xi_\beta(t, \theta, x)\right) \cdot \partial_\vartheta \xi_\beta(t, \theta, x) \\ &+ \partial_x^2 F_\beta\left(t\rho + \theta, \xi_\beta(t, \theta, x)\right) \cdot \left(\partial_\vartheta \xi_\beta(t, \theta, x)\right)^2 \\ &+ \partial_x F_\beta\left(t\rho + \theta, \xi_\beta(t, \theta, x)\right) \cdot \partial_\vartheta^2 \xi_\beta(t, \theta, x). \end{aligned}$$

The solution is straightforwardly given by

$$\begin{aligned}&\partial_\vartheta^2 \xi_\beta(t,\theta,x) = \\ &\int_0^t \Big[\partial_x^2 F_\beta\left(s\rho+\theta, \xi_\beta(s,\theta,x)\right)\left(\partial_\vartheta \xi_\beta(s,\theta,x)\right)^2 + \partial_\vartheta^2 F_\beta\left(s\rho+\theta, \xi_\beta(s,\theta,x)\right) \\ &+ 2\partial_x\partial_\vartheta F_\beta\left(s\rho+\theta, \xi_\beta(s,\theta,x)\right)\partial_\vartheta \xi_\beta(s,\theta,x)\Big] \exp\left(\int_s^t \partial_x F_\beta\left(\tau\rho+\theta, \xi_\beta(\tau,\theta,x)\right) d\tau\right) ds.\end{aligned} \tag{6.1.9}$$

6.1.2. Reduction to a Poincaré map

Let us drop the index β in this paragraph and set $d = D - 1$.

Assume without loss of generality that $|\rho_D| = \max_{j=1,\ldots,D} |\rho_j|$. Note that since $\rho = (\rho_1, \ldots, \rho_D) \in \mathbb{R}^D$ is Diophantine of type $(\mathscr{C}, \eta)$ (in the sense of Definition 3.3.1), we have that $\omega = \omega(\rho) = (\rho_1/\rho_D, \ldots, \rho_d/\rho_D) \in \mathbb{T}^d$ is Diophantine of type $(\mathscr{C}', \eta')$ (in the sense of Definition 3.1.1) with $\eta' = \eta$ and $\mathscr{C}'$ proportional to $\mathscr{C}/\rho_D$.

Before we can reduce the *local* flow Ξ from (3.3.1) to a skew product of the form (3.1.1), we need to make it a flow, that is, we need the set U to equal $\mathbb{R} \times \mathbb{T}^D \times X$ so that any point in $\mathbb{T}^D \times X$ has a full forward orbit. To that end, recall that we are dealing with local bifurcations occurring in a section $\Gamma = \mathbb{T}^D \times [\gamma^-, \gamma^+] \subseteq \mathbb{T}^D \times X$ and assume, for simplicity, that $[\gamma^-, \gamma^+]$ is in the interior of X. By changing the non-autonomous vector field F outside of Γ, we obviously do not change anything about the considered bifurcation within Γ. Hence, we may replace F by $\tilde{F} = h \cdot F$, where $h\colon \Theta \times X \ni (\theta, x) \mapsto \tilde{h}(x) \in [0,1]$ is a smooth function with $\tilde{h}\restriction_{[\gamma^- - \varepsilon, \gamma^+ + \varepsilon]} = 1$ and $\tilde{h}\restriction_{X \setminus [\gamma^- - 2\varepsilon, \gamma^+ + 2\varepsilon]} = 0$ for some $\varepsilon > 0$ with $[\gamma^- - 3\varepsilon, \gamma^+ + 3\varepsilon] \subseteq X$. With $\tilde{F}$, we actually have a flow since a given orbit either stays within $[\gamma^- - 2\varepsilon, \gamma^+ + 2\varepsilon]$ or is eventually constant so that every orbit is well-defined for all times. In the following, we will not stress this detail. However, the reader should always think of the modified vector field $\tilde{F}$ whenever full orbits are assumed for arbitrary initial conditions.

In this sense, consider the first return map to the Poincaré section $\mathbb{T}^d_D = \{\theta \in \mathbb{T}^D \colon \theta_D = 0\}$, that is, the map

$$\begin{aligned}\tilde{\Xi}\colon \mathbb{T}^d_D \times X &\to \mathbb{T}^d_D \times X \\ (\theta, x) &\mapsto \Xi(1/\rho_D, \theta, x) = \left(\theta + 1/\rho_D \cdot \rho, \tilde{\xi}_\theta(x)\right),\end{aligned} \tag{6.1.10}$$

where $\tilde{\xi}_\theta(x) = \xi(1/\rho_D, \theta, x)$. Note that (6.1.10) is of the form (3.1.1). From now on, we identify $\mathbb{T}^d_D$ with $\mathbb{T}^d$ and thus consider $\mathbb{T}^d$ a subset of $\mathbb{T}^D$ (slightly abusing terminology). In a similar fashion, we may write $\theta + \omega$ when $\theta \in \mathbb{T}^D$ and actually $\theta + 1/\rho_D \cdot \rho$ is meant.

It is obvious that an invariant graph of the flow Ξ from (3.3.1) yields an invariant graph for $\tilde{\Xi}$ in (6.1.10). The following basic observation shows how an invariant graph for the first return map $\tilde{\Xi}$ yields the existence of an invariant graph for the flow Ξ and how properties of the one are carried over to the other.

Proposition 6.1.1. *Consider the flow Ξ in (3.3.1) with a non-autonomous C^1-vector field F and suppose there is an invariant graph $\tilde{\phi}\colon \mathbb{T}^d \to X$ for the corresponding first return map $\tilde{\Xi}$. Then there is a unique invariant graph ϕ for Ξ with $\phi(\theta) = \tilde{\phi}(\theta)$ for each $\theta \in \mathbb{T}^d$ and ϕ is continuous if and only if $\tilde{\phi}$ is continuous. Further, if $\tilde{\Phi}$ is relatively compact in $\mathbb{T}^d \times X$, then $\lambda(\phi) = \rho_D \cdot \lambda(\tilde{\phi})$.*

Proof. For $\theta \in \mathbb{T}^D$, set $t_\theta = \theta_D/\rho_D$. Then, $\phi\colon \theta \mapsto \xi\left(t_\theta, \theta - t_\theta\rho, \tilde{\phi}(\theta - t_\theta\rho)\right)$ is invariant under Ξ. The uniqueness and the assertion about the continuity are obvious.

Now, note that if $\overline{\tilde{\Phi}}$ is compact in X, then so is $\overline{\Phi} \subseteq \Xi\left([0,1] \times \overline{\tilde{\Phi}}\right)$. As F is C^1, $\partial_x F$ is thus bounded on $\overline{\Phi}$ and hence integrable. By means of (6.1.5), we therefore have

$$\begin{aligned}
\lambda(\tilde{\phi}) &= \int_{\mathbb{T}^d} \log\left|\partial_x \tilde{\xi}_\theta\left(\tilde{\phi}(\theta)\right)\right| d\theta = \int_{\mathbb{T}^d}\int_0^{1/\rho_D} \partial_x F_\beta\left(\theta + s\rho, \xi_\beta(s, \theta, \tilde{\phi}(\theta)\right) ds\, d\theta \\
&= \int_{\mathbb{T}^d}\int_0^{1/\rho_D} \partial_x F_\beta\left(\theta + s\rho, \phi(\theta + s\rho)\right) ds\, d\theta = 1/\rho_D \cdot \int_{\mathbb{T}^D} \partial_x F_\beta\left(\theta, \phi(\theta)\right) d\theta
\end{aligned}$$

and

$$\begin{aligned}
\lambda(\phi) &= \rho_D \cdot \int_{\mathbb{T}^D} \log\left|\partial_x \xi\left(1/\rho_D, \theta, \phi(\theta)\right)\right| d\theta \\
&= \rho_D \cdot \int_{\mathbb{T}^1}\int_{\mathbb{T}^d}\int_0^{1/\rho_D} \partial_x F_\beta\left(\theta + s\rho, \phi(\theta + s\rho)\right) ds\, d(\theta_1, \ldots, \theta_d)\, d\theta_D \\
&= \int_{\mathbb{T}^1}\int_{\mathbb{T}^D} \partial_x F_\beta\left(\theta, \phi(\theta)\right) d\theta\, d\theta_D = \rho_D \cdot \lambda(\tilde{\phi}).
\end{aligned}$$

□

6.2. The quasiperiodically driven logistic differential equation

We are now in a position to explicitly provide a class of simple examples whose first return maps lie in $\mathcal{U}_\omega(X)$ with $\omega = \omega(\rho)$ as in the previous paragraph. We study a one-parameter family of skew product flows Ξ_β of the form (3.3.1) with $X = \mathbb{R}$ and

$$F_\beta(\theta, x) = -bx^2 + b - \beta b/\left(1 - b^{-1/2}\right) \cdot g(\theta), \qquad (*)$$

where $b > 1$ and $g\colon \mathbb{T}^D \to [0,1]$ is C^2 and assumes a unique non-degenerate global maximum at some $\overline{\theta} \in \mathbb{T}^D$. Without loss of generality, we may assume that $g\left(\overline{\theta}\right) = 1$.

It is not hard to see that $(F_\beta)_{\beta\in[0,1]}$ lies in $\mathscr{P}(\mathbb{R})$ and satisfies (2.1.16)–(2.1.19) with $\gamma^+ = 1$ and $\gamma^- = -1$ (where (2.1.19) follows from Claim 6.2.2 below), that is, $(*)$ undergoes a saddle-node bifurcation in the sense of Theorem 2.1.25. Our goal is to show that for sufficiently large b, $(*)$ undergoes a *non-smooth* bifurcation (cf. Figure 6.1).

By the above, we hence have to check one by one if the hypothesis $(\mathscr{A}1)$–$(\mathscr{A}15)$ apply to the first return maps $(\tilde{\xi}_\beta)_{\beta\in[0,1]}$ corresponding to $(*)$.

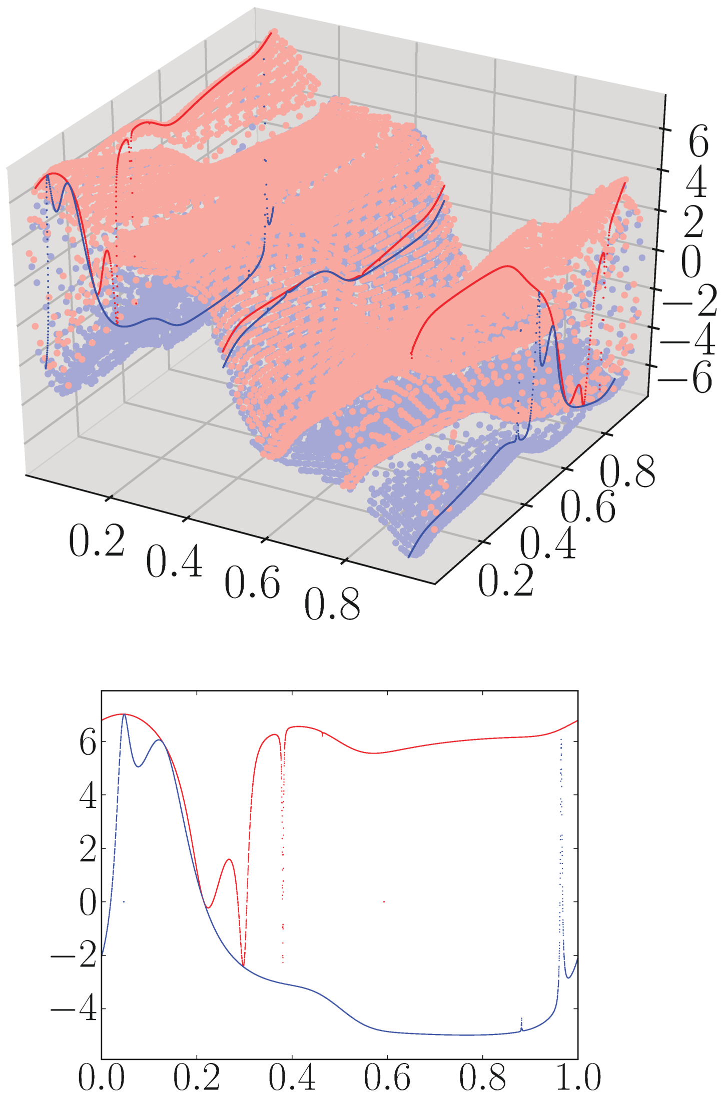

Figure 6.1. Invariant graphs of a skew product flow with $D = 2$ close to a non-smooth saddle-node bifurcation. The considered family of vector fields is given by $F_\beta(\theta, x) = -x^2 + b - \beta \cdot (2 - \cos^{11}(2\pi\theta_1) - \cos^{11}(2\pi\theta_2))/4$ where $\theta = (\theta_1, \theta_2)$. We put $\rho = ((\sqrt{5} - 1)/2, \pi)$, $b = 100$, and $\beta = 176.01538$. In the upper picture, the attractor is pale red, the repeller is depicted in pale blue. The curves in deep red and blue show slices of the attractor and repeller, respectively, for three different fixed values of θ_1. The lower picture allows a closer look at the section close to $\theta_1 = 0$.

We would like to remark that studying $(*)$ amounts to studying the quasiperiodically forced logistic differential equation whose non-autonomous vector field is given by

$$L_\beta(\theta, x) = 2/r \cdot bx \cdot (r - x) - \beta b/(1 - b^{-1/2}) \cdot g(\theta), \tag{$**$}$$

for some $r > 0$. To see this, note that solutions χ_β of

$$\partial_t \chi_\beta(t, \theta, x) = L_\beta(\rho_t(\theta), \chi_\beta(t, \theta, x))$$

with $\chi_\beta(0, \theta, x) = x$ yield solutions $\xi_\beta(t, \theta, x) = 2/r \cdot \chi_\beta[t, \theta, r/2 \cdot (x + 1)] - 1$ of (6.1.2). In particular, this shows that every invariant graph ψ_β of L_β corresponds to an invariant graph $\phi_\beta(\theta) = 2/r \cdot \psi_\beta(\theta) - 1$ of F_β and vice-versa. Since ψ_β is continuous if and only if ϕ_β is, we immediately get that $(**)$ undergoes a non-smooth saddle-node bifurcation if and only if $(*)$ does.

To reduce the technicalities of our investigation to a minimum, we assume without further mentioning that $g(\theta) = h(|\theta - \overline{\theta}|)$, where h is a non-increasing C^2-bump-function $h\colon \mathbb{R}_{\geq 0} \to [0, 1]$ with $h'(0) = 0$, $h'\restriction_{(0,R)} < 0$ (for some $R > 0$), $h''(0) < 0$, $h(y) = 0$ for all $y \geq R$, and $h(0) = 1$.

Let us introduce some notation. Given θ and θ' in $\mathbb{T}^D$ such that there is $s \in [0, 1/\rho_D]$ with $\theta' = \theta + s\rho$, we denote by $[\theta, \theta']$ the line segment $\{\theta + \tau\rho \colon \tau \in [0, s]\}$. Similarly, given $A, B \subseteq \mathbb{T}^D$ such that for all $\theta \in A$ there exists a unique $s(\theta) \in [0, 1/\rho_D]$ so that $B = \{\theta + s(\theta)\rho \colon \theta \in A\}$, we set $[A, B] = \bigcup_{\theta \in A} [\theta, \theta + s(\theta)\rho]$.

We suppose there is $\delta_1 > 0$ much smaller than $1/\rho_D$ (it suffices to assume $\delta_1 < \min\{1/18, 1/(36\rho_D)\}$) such that $[\mathbb{T}^d, \mathbb{T}^d - \delta_1\rho] \cap B_R\left(\overline{\theta}\right) = \emptyset$, that is, in one iteration, the time span before an orbit hits the bump is much bigger than the time after hitting the bump. Without loss of generality, we may further assume there is a positive constant $\delta_2 < \delta_1$ such that $[\mathbb{T}^d - \delta_2\rho, \mathbb{T}^d] \cap B_R\left(\overline{\theta}\right) = \emptyset$. By possibly shifting the $\theta_D = 0$ section, both assumptions boil down to assuming that R is small (but still fixed, independently of b).

For the convenience of the reader, we repeat the assumptions ($\mathcal{A}$1)–($\mathcal{A}$15) in terms of the maps $\tilde{\xi}_{\beta,\theta}$. To that end, we set $C = [1 - c, 1 + c]$ for some positive $c < 1/4$ and $E = [-1, -1 + \exp(-b/(2\rho_D))]$ and restrict our analysis to the section $\Gamma = \mathbb{T}^d \times [-1, 1 + c]$. Recall that the assumptions below are supposed to hold for all $\beta \in [\beta_-(0), \beta_+(0)]$ (if applicable), where–as before–we put

$$\beta_-(0) = \min\{\beta \in [0, 1] \colon \exists \theta \in \mathbb{T}^d \text{ such that } \tilde{\xi}_{\beta,\theta}(1 - c) \leq -1 + \exp(-b/[2\rho_D])\},$$
$$\beta_+(0) = \max\{\beta \in [0, 1] \colon \tilde{\xi}_{\beta,\theta}(1 + c) \geq -1 \text{ for all } \theta \in \mathbb{T}^d\}.$$

For the sake of readability, we set $\beta_\pm = \beta_\pm(0)$. As in the discrete time case, the well-definition of $\beta_\pm$ will follow from ($\mathcal{A}$7). Finally, we define

$$\mathcal{I}_0 = \left\{\theta \in \mathbb{T}^d \colon [\theta, \theta + \omega] \cap \overline{B_R\left(\overline{\theta}\right)} \neq \emptyset\right\}$$

and let

$$\alpha_c^{-1} = \alpha_e = \exp\left[2b(1-c)(1/\rho_D - \delta_1) - 10b\delta_1\right] \quad \text{and} \quad \alpha_l^{-1} = \alpha_u = \exp[2b(1+c)/\rho_D].$$

With these definitions, $(\mathcal{A}1)$–$(\mathcal{A}8)$ adapted to $\tilde{\Xi}_\beta$ read

($\mathcal{A}1$) $0 < \partial_x\tilde{\xi}_{\beta,\theta}(x) < \alpha_c$ for $(\theta, x) \in \mathbb{T}^d \times C$;

($\mathcal{A}2$) $\partial_x\tilde{\xi}_{\beta,\theta}(x) > \alpha_e$ for $(\theta, x) \in (\mathbb{T}^d \setminus \mathcal{I}_0) \times E \bigcap \tilde{\Xi}_\beta^{-1}(\mathbb{T}^d \times E)$;

($\mathcal{A}3$) $\alpha_l < \partial_x\tilde{\xi}_{\beta,\theta}(x) < \alpha_u$ for all $(\theta, x) \in \Gamma \cap \tilde{\Xi}_\beta^{-1}(\Gamma)$;

($\mathcal{A}4$) $\tilde{\xi}_{\beta,\theta}(1+c) \le 1+c$ and $\tilde{\xi}_{\beta,\theta}(-1) \le -1$;

($\mathcal{A}5$) $\tilde{\xi}_{\beta,\theta}(x) \in C$ for all $x \in [-1 + \exp(-b/(2\rho_D)), 1+c]$ and $\theta \notin \mathcal{I}_0$.

There is a set $\mathcal{J}_{0,\beta} \subseteq \mathcal{I}_0$ which contains (at least) all θ for which $\tilde{\xi}_{\beta,\theta}(1-c) \le -1 + \exp(-b/(2\rho_D))$ and

($\mathcal{A}6$) $\mathcal{J}_{0,\beta}$ is closed and convex and $\mathcal{J}_{0,\beta} \subseteq \mathcal{J}_{0,\beta'}$ for each $\beta' \ge \beta$.

($\mathcal{A}7$) $\tilde{\xi}_{0,\theta}(1-c) \ge 1-c$ for all $\theta \in \mathbb{T}^d$ and $\tilde{\xi}_{\beta_+,\theta}(1+c) \le -1$ for some $\theta \in \mathbb{T}^d$;

($\mathcal{A}8$) $\tilde{\xi}_{(\cdot)}(\theta, x)$ is non-increasing for fixed $(\theta, x) \in \Gamma$.

Before we come to prove $(\mathcal{A}1)$–$(\mathcal{A}8)$, we provide some simple observations. From now on, given the rotation vector ρ, we denote by $\theta_0 \in \mathbb{T}^d$ that point which passes through the maximum in $\overline{\theta}$ within one time step, that is, $\overline{\theta} \in [\theta_0, \theta_0 + \omega]$.

Proposition 6.2.1. *Suppose F_β is given by $(*)$. Then*

(a) $\xi_\beta(t, \theta, x) < x$ if $|x| > 1$ where $t > 0$, $\theta \in \mathbb{T}^D$, and $\beta \in [0, 1]$.

(b) $\xi_\beta(t, \theta, x) \ge x$ if $|x| \le 1$ where $t \in [0, 1/\rho_D]$ is such that $[\theta, \theta + t\rho] \cap B_R\left(\overline{\theta}\right) = \emptyset$, $\theta \in \mathbb{T}^D$, and $\beta \in [0, 1]$.

Proof. (a) follows easily from the fact that $F_\beta(\theta, x) < 0$ for arbitrary β, θ, and $|x| > 1$. Likewise, (b) follows from the fact that $F_\beta(\theta, x) \ge 0$ for arbitrary β, $\theta \notin B_R\left(\overline{\theta}\right)$, and $|x| \le 1$. □

Now, let us consider $(\mathcal{A}1)$–$(\mathcal{A}8)$ in opposite order. $(\mathcal{A}8)$ follows immediately from the monotone dependence of $(*)$ on β. The first part of $(\mathcal{A}7)$ is immediate. The existence of $\beta_+ \in (0, 1)$ such that the second estimate of $(\mathcal{A}7)$ holds true follows from the next statement under the assumption of sufficiently large b.

It is convenient to introduce $U_\varepsilon = \{\theta \in \mathbb{T}^D : g(\theta) \ge 1 - \varepsilon\}$ where $\varepsilon > 0$. Clearly, U_ε is nothing but $B_{h^{-1}([1-\varepsilon,1])}\left(\overline{\theta}\right)$.

Claim 6.2.2. *Suppose F_β is given by $(*)$ and b is sufficiently large. Then there exists $\beta \in (0,1)$ such that $\xi_\beta(t,\theta_0,1+c)$ is well-defined for all $t \in [0,1/\rho_D]$ and $\tilde{\xi}_{\beta,\theta_0}(1+c) \le -1$.*

Proof. Note that for t with $\theta_0 + t\rho \in U_{b^{-1/2}/2}$, we have $\partial_t \xi_\beta(t,\theta_0,1+c) = -b\xi_\beta(t,\theta_0,1+c)^2 + b - \beta b/(1-b^{-1/2})g(\theta+t\rho) \le -b\xi_\beta(t,\theta_0,1+c)^2 - b(\beta/(2b^{1/2}-2)+\beta-1)$. Consider such β for which $\beta/(2b^{1/2}-2)+\beta-1 > 0$. Observe that

$$y_\beta(t) = -\sqrt{\beta/(2b^{1/2}-2)+\beta-1}\,\tan\left(b\sqrt{\beta/(2b^{1/2}-2)+\beta-1}(t-t_0)+\alpha\right)$$

with $\alpha = \arctan\left(-(1+c)/\sqrt{\beta/(2b^{1/2}-2)+\beta-1}\right)$ solves

$$\partial_t y_\beta(t) = -by_\beta(t)^2 - b(\beta/(2b^{1/2}-2)+\beta-1)$$

with $y_\beta(t_0) = 1+c$. Thus, $y_\beta(t)$ is an upper bound for $\xi_\beta(t,\theta_0,1+c)$ for all $t \in [t_0,t_1]$, where $[t_0,t_1]$ is set to be the maximal interval with $[\theta_0+t_0\rho,\theta_0+t_1\rho] \subseteq U_{b^{-1/2}/2}$. Note that $|t_1-t_0| > b^{-1/2}$ for big enough b since g assumes a minimum in $\overline{\theta}$. Now for large enough b, there is $\beta \in (0,1)$ such that $y_\beta(b^{-1/2}+t_0) < -1$ which proves that the image of $[0,1) \ni \beta \mapsto \xi_\beta(t_1,\theta_0,1+c)$ contains $[-1,1]$. Proposition 6.2.1(a) and the continuous dependence of $\tilde{\xi}_{\beta,\theta_0}(1+c)$ on β hence yield the statement. □

$(\mathcal{A}6)$ will be treated in Lemma 6.2.5. For sufficiently large b, $(\mathcal{A}5)$ is a consequence of the following statement.

Claim 6.2.3. *Suppose F_β is given by $(*)$, $\theta \notin \mathcal{I}_0$ and b is sufficiently large. Then $\tilde{\xi}_{\beta,\theta}(-1+\exp[-b/(2\rho_D)]) > 1-c$.*

Proof. Note that as $\theta \notin \mathcal{I}_0$ we have that $\xi_\beta(t,\theta,-1+\exp(-b/(2\rho_D)))$ equals $y(t)$ for $t \in [0,1/\rho_D]$, where y is the solution of the initial value problem

$$\dot{y} = -by^2 + b, \quad y(0) = -1+\exp(-b/(2\rho_D)).$$

Now, $y(t) = \tanh(b\cdot t+\alpha)$, where $|\alpha| = |\operatorname{artanh}(-1+\exp[-b/(2\rho_D)])| \le b/(3\rho_D)$. Hence, $y(1/\rho_D) \ge \tanh(2b/(3\rho_D)) > 1-c$ for large enough b. □

$(\mathcal{A}4)$ is an immediate consequence of Proposition 6.2.1 (a). Hence, apart from $(\mathcal{A}6)$, we are left with $(\mathcal{A}1)$–$(\mathcal{A}3)$ which follow from the next assertion.

Proposition 6.2.4. *Suppose F_β is given by $(*)$ and b is sufficiently large. Then*

(a) $\partial_x\tilde{\xi}_{\beta,\theta}(x) \le \exp(-2b(1-c)(1/\rho_D-\delta_1)+4b\delta_1)$ *and* $\tilde{\xi}_{\beta,\theta}(x) \ge -2$ *for* $(\theta,x) \in \mathbb{T}^d \times C$*;*

(b) $\partial_x\tilde{\xi}_{\beta,\theta}(x) \ge \exp(2b(1-\exp[-b/(2\rho_D)])/\rho_D)$ *for* $(\theta,x) \in (\mathbb{T}^d \setminus \mathcal{I}_0)\times E \cap \tilde{\Xi}_\beta^{-1}(\mathbb{T}^d \times E)$*;*

(c) $\exp(-2b(1+c)/\rho_D) < \partial_x\tilde{\xi}_{\beta,\theta}(x) \le \exp(2b/\rho_D)$ *for all* $(\theta,x) \in \Gamma \cap \tilde{\Xi}_\beta^{-1}(\Gamma)$*.*

Proof. Note that due to Proposition 6.2.1(a), we have that $(\theta, x) \in \Gamma \cap \tilde{\Xi}_\beta^{-1}(\Gamma)$ necessarily implies $\xi_\beta(t, \theta, x) \in [-1, 1 + c)$ for all $t \in (0, 1/\rho_D]$.

Now, (c) follows from equation (6.1.5) since we have $2b \geq \partial_x F_\beta > -2b(1 + c)$ on $\mathbb{T}^D \times [-1, 1 + c)$. Similarly, we obviously get item (b) as long as $\xi_\beta(t, \theta, x) \in E$ for all $t \in [0, 1/\rho_D]$ which necessarily holds for $(\theta, x) \in (\mathbb{T}^d \setminus \mathcal{I}_0) \times E \bigcap \tilde{\Xi}_\beta^{-1}(\mathbb{T}^d \times E)$ due to Proposition 6.2.1 (b).

It remains to consider (a). Note that for all $x \in C$, all $\beta \in [0, 1]$, each $\theta \in \mathbb{T}^d$, and $t \in [0, 1/\rho_D - \delta_1]$ we have $\xi_\beta(t, \theta, x) \geq \xi_\beta(t, \theta, 1 - c) \geq 1 - c$. Suppose there was $\theta' \in \mathbb{T}^d$ and $\beta \in [0, \beta_+]$ such that $\tilde{\xi}_{\beta,\theta'}(1 - c) = -2$. Note that in this case $\xi_\beta(t, \theta', 1 - c) \geq -2$ holds necessarily for all $t \in [0, 1/\rho_D]$ because of Proposition 6.2.1(a). Thus, (6.1.5) yields

$$
\begin{aligned}
\partial_x \tilde{\xi}_{\beta,\theta'}(x) &= \exp\left(\int_0^{1/\rho_D} \partial_x F_\beta(s\rho + \theta', \xi_\beta(s, \theta', x))\, ds\right) \\
&\leq \exp\left(-2b \int_0^{1/\rho_D - \delta_1} \xi_\beta(s, \theta', x)\, ds - 2b \int_{1/\rho_D - \delta_1}^{1/\rho_D} \xi_\beta(s, \theta', x)\, ds\right) \qquad (6.2.1) \\
&\leq \exp(-2b(1 - c)(1/\rho_D - \delta_1) + 4b\delta_1)
\end{aligned}
$$

for all $x \in [1 - c, 1 + c]$. Hence in this case, we had $\tilde{\xi}_{\beta_+,\theta'}(1 + c) \leq \tilde{\xi}_{\beta,\theta'}(1 + c) \leq -2 + 2c \cdot \exp(-2b(1 - c)(1/\rho_D - \delta_1) + 4b\delta_1) < -1$ (where the last inequality holds for large enough b if we assume $\delta_1 < 1/(36\rho_D)$) contradicting the definition of β_+. Thus, we have $\tilde{\xi}_{\beta,\theta}(x) > -2$ for all $\theta \in \mathbb{T}^d$, $x \in C$ and $\beta \in [0, \beta_+]$ and hence (6.2.1) yields an upper bound for $\partial_x \tilde{\xi}_{\beta,\theta}(x)$ with $(\theta, x) \in \mathbb{T}^d \times C$. □

The main work of this chapter is to show that there is $\mathcal{J}_{0,\beta}$ and $s > 0$ such that ($\mathcal{A}$6) holds and

($\mathcal{A}$9) $\partial_\vartheta^2 \tilde{\xi}_{\beta,\theta}(x) > s$ for each $\vartheta \in \mathbb{S}^{d-1}$, $x \in C$ and all $\theta \in \mathcal{J}_{0,\beta}$.

Lemma 6.2.5. *Suppose F_β is given by ($*$), b is sufficiently large, and $\beta \in [\beta_-, \beta_+]$.*

Then there is $\mathcal{J}_{0,\beta} \subseteq \mathcal{I}_0$ such that $\partial_\vartheta^2 \tilde{\xi}_{\beta,\theta}(x) > \exp(b\delta_2/4)$ for arbitrary $x \in C$, $\vartheta \in \mathbb{S}^{d-1}$, and $\theta \in \mathcal{J}_{0,\beta}$. Further, $\mathcal{J}_{0,\beta}$ contains all $\theta \in \mathbb{T}^d$ with $\tilde{\xi}_{\beta,\theta}(1 - c) \leq -1 + \exp(-b/(2\rho_D))$ and ($\mathcal{A}$6) is satisfied.

For later use, we provide some crude and straightforward estimates in the following auxiliary statement. We denote by $\mathbf{1}_A$ the characteristic function of a set $A \subseteq \mathbb{T}^D$, that is, $\mathbf{1}_A = 1$ on A and $\mathbf{1}_A = 0$ on $\mathbb{T}^D \setminus A$.

Claim 6.2.6. *For $(\theta, x) \in \mathbb{T}^d \times C$, $\beta \in [0, \beta_+]$, $t_1 \in [0, 1/\rho_D - \delta_1]$, and $t \in [t_1, 1/\rho_D]$, we have under the assumption of sufficiently large b that*

$$\int_0^{t-t_1} \exp\left(\int_s^{t-t_1} \partial_x F_\beta \left((\tau + t_1)\rho + \theta, \xi_\beta(\tau + t_1, \theta, x)\right) d\tau\right) ds \leq \exp(5b\delta_1). \tag{6.2.2}$$

Further, suppose $\tilde{\xi}_{\beta,\theta}(x) \leq -3/4$ and $t \geq 1/\rho_D - \delta_2/2$. There is $R_0 < R$ such that for sufficiently large b

$$\int_0^{t-t_1} \mathbf{1}_{B_{R_0}(\overline{\theta})}((s + t_1)\rho + \theta) \exp\left(\int_s^{t-t_1} \partial_x F_\beta \left((\tau + t_1)\rho + \theta, \xi_\beta(\tau + t_1, \theta, x)\right) d\tau\right) ds$$
$$\geq \exp(b\delta_2/2). \tag{6.2.3}$$

Finally, if $0 \leq t_1 \leq 1/\rho_D - 5\delta_1$ and $t \in [t_1, 1/\rho_D]$, then we have for all $(\theta, x) \in \mathbb{T}^d \times C$ and sufficiently large b that

$$\exp\left(\int_0^{t-t_1} \partial_x F_\beta \left((\tau + t_1)\rho + \theta, \xi_\beta(\tau + t_1, \theta, x)\right) d\tau\right) \leq 1. \tag{6.2.4}$$

Proof of the claim. The relations can be seen in a similar fashion as (6.2.1). In particular, we make use of the fact that $\xi_\beta(\tau + t_1, \theta, x) \geq -2$ for all $\tau \in [0, 1/\rho_D - t_1] \supseteq [0, t - t_1]$ and $\xi_\beta(\tau + t_1, \theta, x) \geq 1 - c$ for $\tau \leq 1/\rho_D - \delta_1 - t_1$. For $x \in C$, this implies

$$\exp\left(\int_s^{t-t_1} \partial_x F_\beta \left((\tau + t_1)\rho + \theta, \xi_\beta(\tau + t_1, \theta, x)\right) d\tau\right) = \exp\left(-2b \int_s^{t-t_1} \xi_\beta(\tau + t_1, \theta, x)\, d\tau\right)$$
$$\leq \exp(4b\delta_1)$$

such that

$$\int_0^{t-t_1} \exp\left(\int_s^{t-t_1} \partial_x F_\beta \left((\tau + t_1)\rho + \theta, \xi_\beta(\tau + t_1, \theta, x)\right) d\tau\right) ds \leq 1/\rho_D \cdot \exp(4b\delta_1),$$

which is smaller than $\exp(5b\delta_1)$ for big enough b.

For the second inequality, note that there is $0 < \tilde{R} < R$ such that for big enough b we have $F_\beta(\theta, -3/4) \geq 0$ for all $\theta \notin B_{\tilde{R}}\left(\overline{\theta}\right)$ and all β. Hence, for all θ and x as in the assumptions we necessarily have that $\xi_\beta(\tau, \theta, x) \leq -3/4$ for all $\tau \in [0, 1/\rho_D]$ with $[\theta + \tau\rho, \theta + \omega] \cap B_{\tilde{R}}\left(\overline{\theta}\right) = \emptyset$. Set $R_0 = (R + \tilde{R})/2$. Then,

$$\int_0^{t-t_1} \mathbf{1}_{B_{R_0}(\overline{\theta})}((s + t_1)\rho + \theta) \exp\left(\int_s^{t-t_1} \partial_x F_\beta \left((\tau + t_1)\rho + \theta, \xi_\beta(\tau + t_1, \theta, x)\right) d\tau\right) ds$$
$$\geq \int_0^{t-t_1} \mathbf{1}_{B_{R_0}(\overline{\theta}) \setminus B_{\tilde{R}}(\overline{\theta})}((s + t_1)\rho + \theta) \exp\left(\int_s^{t-t_1} \partial_x F_\beta \left((\tau + t_1)\rho + \theta, \xi_\beta(\tau + t_1, \theta, x)\right) d\tau\right) ds$$
$$\geq \left(R_0 - \tilde{R}\right)/\rho_D \cdot \exp\left(-2b \int_{1/\rho_D - \delta_2 - t_1}^{t-t_1} \xi_\beta(\tau + t_1, \theta, x)\, d\tau\right) \geq \left(R_0 - \tilde{R}\right)/\rho_D \cdot \exp\left(3/4 \cdot b\delta_2\right)$$

which is clearly bigger than $\exp(b\delta_2/2)$ for large enough b.

For the last relation, note that since $t_1 \leq 1/\rho_D - 5\delta_1$, we have for $t \geq 1/\rho_D - \delta_1$ that

$$\begin{aligned}
&\int_0^{t-t_1} \partial_x F_\beta\Big((\tau + t_1)\rho + \theta, \xi_\beta(\tau + t_1, \theta, x)\Big)\, d\tau \\
&= \int_0^{1/\rho_D-\delta_1-t_1} \partial_x F_\beta\Big((\tau + t_1)\rho + \theta, \xi_\beta(\tau + t_1, \theta, x)\Big)\, d\tau \\
&\quad + \int_{1/\rho_D-\delta_1-t_1}^{t-t_1} \partial_x F_\beta\Big((\tau + t_1)\rho + \theta, \xi_\beta(\tau + t_1, \theta, x)\Big)\, d\tau \leq -8b\delta_1 \cdot (1-c) + 4b\delta_1 < 0
\end{aligned}$$

since $c < 1/4$. Note that if $t < 1/\rho_D - \delta_1$, then (6.2.4) is obvious. □

Proof of Lemma 6.2.5. Let us fix some notation. For the rest of this proof, ϑ is always assumed to be some element of $\mathbb{R}^d$ (the tangent space of $\mathbb{T}^d$ at any $\theta \in \mathbb{T}^d$) with $|\vartheta| = 1$. In contrast, Δ and Δ' always denote elements of $\mathbb{S}^d \subseteq \mathbb{R}^D$ orthogonal to ρ in the following. Set $T_\tau = \{\theta_0 + \tau\rho + \varepsilon\Delta\colon \Delta \perp \rho,\ |\Delta| = 1 \text{ and } |\varepsilon| \leq R\}$ where $\tau \in (0, 1/\rho_D)$, that is, T_τ is a d-dimensional disk of radius R orthogonal to ρ, centred at $\theta_0 + \tau\rho$. Similarly, set $\tilde{T}_\tau = \{\theta_0 + \tau\rho + \varepsilon\Delta\colon \Delta \perp \rho,\ |\Delta| = 1 \text{ and } |\varepsilon| \leq r_b\}$ where $r_b = \exp(-9b\delta_1)$. We denote by $P\theta$ the *orthogonal projection* of $\theta \in \mathbb{T}^D$ onto $[\theta_0, \theta_0 + \omega]$ so that $\theta = P\theta + \varepsilon\Delta$, where $|\varepsilon|$ is minimal (with $\Delta \perp \rho$ and $|\Delta| = 1$ as above). Set $t_1 = 1/(4\rho_D)$ and note that–if R is small enough–$T_{t_1} \cap \mathbb{T}^d = \emptyset$ and $[\mathcal{I}_0, T_{t_1}] \cap B_R\big(\overline{\theta}\big) = \emptyset$. Let t_2 be such that T_{t_2} has a positive distance to $B_R\big(\overline{\theta}\big)$ and such that $T_{t_2} \cap \mathbb{T}^d = \emptyset$ and $[\mathcal{I}_0, T_{t_2}] \cap B_R\big(\overline{\theta}\big) = B_R\big(\overline{\theta}\big)$. Again, we might have to assume small enough R in order for such t_2 to exist. Finally, $t_3 > t_2$ with $\tilde{T}_{t_3} \cap \mathbb{T}^d = \emptyset$ will be chosen to be close to $1/\rho_D$ so that within one iteration, orbits starting in $\tilde{\mathcal{I}}_0 = \{\theta \in \mathbb{T}^d\colon [\theta, \theta + \omega] \cap \tilde{T}_{t_1} \neq \emptyset\}$ enter and leave the bump between $\tilde{T}_{t_1}$ and $\tilde{T}_{t_3}$ while the remaining time between $\tilde{T}_{t_3}$ and $\mathbb{T}^d$ will be negligibly short. We let $t_i(\theta) \in [0, 1/\rho_D]$ be such that $\theta + t_i(\theta)\rho \in T_{t_i}$ for $i = 1, 2$ and $\theta \in \mathcal{I}_0$. Likewise for $\theta \in \tilde{\mathcal{I}}_0$, we denote by $t_3(\theta) \in [0, 1/\rho_D]$ that time for which $\theta + t_3(\theta)\rho \in \tilde{T}_{t_3}$. By considering small enough R, we may assume without loss of generality that $t_2(\theta) > 1/\rho_D - \delta_2/2$ for each $\theta \in \tilde{\mathcal{I}}_0$. Note that the t_i are (restrictions of) affine linear maps whose derivatives are given by a constant matrix whose norm we denote by κ (note that obviously $\mathrm{d}t_1(\theta) = \mathrm{d}t_2(\theta) = \mathrm{d}t_3(\theta)$, where d denotes the total derivative).

The Hessian of $g(\theta) = h(|\theta - \overline{\theta}|)$ is easily seen to be

$$\begin{aligned}
\mathrm{d}^2 g(\theta) &= \mathrm{d}\left(\frac{h'(|\theta - \overline{\theta}|)}{|\theta - \overline{\theta}|}(\theta - \overline{\theta})\right) \\
&= \left(\frac{h''(|\theta - \overline{\theta}|)}{|\theta - \overline{\theta}|^2} - \frac{h'(|\theta - \overline{\theta}|)}{|\theta - \overline{\theta}|^3}\right)(\theta - \overline{\theta}) \cdot (\theta - \overline{\theta})^\top + \frac{h'(|\theta - \overline{\theta}|)}{|\theta - \overline{\theta}|} I_D,
\end{aligned}$$

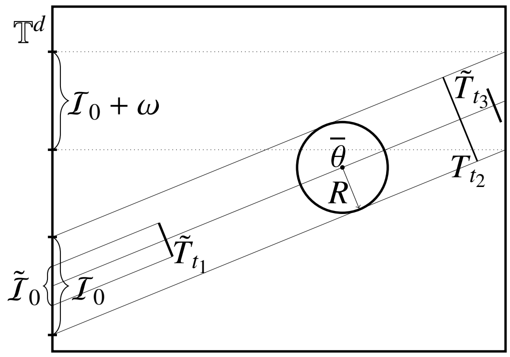

Figure 6.2. The base space for $D = 2$. We subdivide one iteration into three subsequent iterations: first, from $\tilde{\mathcal{I}}_0 \subseteq \mathbb{T}^d$ to $\tilde{T}_{t_1}$. Then, further to $\tilde{T}_{t_3}$. Finally, from $\tilde{T}_{t_3}$ to $\tilde{\mathcal{I}}_0 + \omega \subseteq \mathbb{T}^d$. If $\theta_0 + \tau\rho \in B_R(\overline{\theta})$, the disks $\tilde{T}_\tau$ are sections of the tangents of the level sets of g at $\theta_0 + \tau\rho$.

where I_D denotes the D-dimensional unit matrix. Hence for $\theta = P\theta + \varepsilon\Delta'$, we have

$$\partial_\Delta g(\theta) = \varepsilon \cdot \frac{h'(|\theta - \overline{\theta}|)}{|\theta - \overline{\theta}|}\langle \Delta', \Delta\rangle \quad \text{and} \tag{6.2.5}$$

$$\partial^2_\Delta g(\theta) = \varepsilon^2 \cdot \left(\frac{h''(|\theta - \overline{\theta}|)}{|\theta - \overline{\theta}|^2} - \frac{h'(|\theta - \overline{\theta}|)}{|\theta - \overline{\theta}|^3}\right)\langle \Delta', \Delta\rangle^2 + \frac{h'(|\theta - \overline{\theta}|)}{|\theta - \overline{\theta}|}, \tag{6.2.6}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^D$.

Having in mind (6.1.9), we see that in order to show the positivity of the second derivatives of ξ_β with respect to $\vartheta \in \mathbb{S}^{d-1}$, we need small enough upper bounds on the respective first derivatives in order to ensure that the leading term under the integral is the one containing $\partial^2_\vartheta F_\beta$. To that end, we divide the iteration of an orbit starting at $(\theta, x) \in \tilde{\mathcal{I}}_0 \times C$ into three time intervals (see Figure 6.2). Within the first interval, $[0, t_1(\theta)]$, variation with respect to θ only occurs due to the θ-dependence of $t_1(\theta)$ which turns out to be negligible. The last time interval, $[t_3(\theta), 1/\rho_D]$, will turn out to be negligible as we can assume its length to be small. For the intermediate time interval $[t_1(\theta), t_3(\theta)]$, the crucial point is that by the choice of the sets $\tilde{T}_\tau$ perpendicular to ρ and hence parallel to the level sets of g at the point $\theta_0 + \tau\rho$, the derivatives with respect to ϑ become derivatives with respect to some $\Delta \perp \rho$. By (6.2.5), this implies that in a distance $\varepsilon = r_b$ of $\theta_0 + \tau\rho$ (where $\tau \in [t_1(\theta), t_3(\theta)]$), these derivatives are exponentially small in b (recall that $r_b = \exp[-9b\delta_1]$).

While the first derivatives with respect to ϑ are thus negligible, we will show in Claim 6.2.8 that $(\partial^2_\Delta \xi_\beta)(\tau - t_1, \theta + t_1(\theta)\rho, \xi_\beta(t_1(\theta), \theta, x))$ is bounded away from 0 for each $\tau \in [t_2, t_3]$, provided $\xi_\beta(1/\rho_D, \theta, x)$ is not too far from E. In conclusion, we will show that $\partial^2_\vartheta \xi_\beta(1/\rho_D, \theta, x)$ is bounded away from 0. By means of the next claim, this will finish the proof of Lemma 6.2.5.

Claim 6.2.7. *Consider $\beta \in [\beta_-, \beta_+]$ and assume sufficiently large b. Suppose there is $c_0 > 0$ (independent of b) such that $(\partial^2_\Delta \xi_\beta)(t_2 - t_1, \theta + t_1(\theta)\rho, \xi_\beta(t_1(\theta), \theta, x)) > c_0$ for all $\theta \in \tilde{\mathcal{I}}_0$ and $x \in C$ with $\tilde{\xi}_{\beta,\theta}(x) \leq -3/4$.*

Then there is a closed and convex set $\mathcal{J}_{0,\beta} \subseteq \tilde{\mathcal{I}}_0$ such that $\tilde{\xi}_{\beta,\theta}(1-c) > -1 + \exp(-b/(2\rho_D))$ if $\theta \notin \mathcal{J}_{0,\beta}$. Further, $\tilde{\xi}_{\beta,\theta}(x) \leq -3/4$ for each $\theta \in \mathcal{J}_{0,\beta}$ and $x \in C$, and $\mathcal{J}_{0,\beta} \subseteq \mathcal{J}_{0,\beta'}$ for $\beta \leq \beta' \in [\beta_-, \beta_+]$.

Proof of the claim. For the rest of this proof, given $\theta \in T_{t_2}$, we denote by θ' that point in $\mathcal{I}_0$ for which $\theta = \theta' + t_2(\theta')\rho$.

Observe that the map $u\colon T_{t_2} \ni \theta \mapsto \xi_\beta(t_2(\theta'), \theta', 1)$–we keep the dependence of u on β implicit–assumes its minimum in $\theta_0 + t_2\rho$ and moreover satisfies

$$u(\theta) = \hat{u}(|\theta - (\theta_0 + t_2\rho)|), \tag{6.2.7}$$

where $\hat{u}\colon [0, R] \to X$ is some non-decreasing function. This can be seen as follows: First, we see that $\xi_\beta(t_1(\theta'), \theta', 1) = 1$ for each $\theta' \in \mathcal{I}_0$ since $F_\beta(\theta' + \tau\rho, 1) = 0$ for all $\tau \in [0, t_1(\theta')]$ by definition of t_1. Hence, $u(\theta) = \xi_\beta(t_2 - t_1, \theta - (t_2 - t_1)\rho, 1)$. Now note that for $\tau \in [0, t_2 - t_1]$, we have

$$\begin{aligned}|\theta - (t_2 - t_1)\rho + \tau\rho - \overline{\theta}|^2 &= |\theta - (t_2 - t_1)\rho + \tau\rho - (\theta_0 + (t_1 + \tau)\rho)|^2 + |\theta_0 + (t_1 + \tau)\rho - \overline{\theta}|^2 \\ &= |\theta - (\theta_0 + t_2\rho)|^2 + |\theta_0 + (t_1 + \tau)\rho - \overline{\theta}|^2.\end{aligned}$$

Since $g(\cdot) = h(|(\cdot) - \overline{\theta}|)$, we therefore have that there is $\hat{F}\colon T_{t_2} \times [0, t_2 - t_1] \times X \to \mathbb{R}$ with $F_\beta(\theta - (t_2 - t_1)\rho + \tau\rho, x) = \hat{F}(|\theta - (\theta_0 + t_2\rho)|, \tau, x)$, where $\hat{F}$ is non-decreasing in the first coordinate. This proves (6.2.7).

Set

$$\mathcal{J}_{0,\beta} = \{\theta' \in \mathcal{I}_0 \colon u(\theta) \leq -1 + \exp(-b/(2\rho_D)) + 1/2 \cdot \exp(-b/\rho_D)\}.$$

Obviously, $\mathcal{J}_{0,\beta}$ is closed and $\mathcal{J}_{0,\beta} \subseteq \mathcal{J}_{0,\beta'}$ for $\beta' \geq \beta$. The convexity of $\mathcal{J}_{0,\beta}$ follows from (6.2.7). It hence remains to show that for sufficiently large b we have $\mathcal{J}_{0,\beta} \subseteq \tilde{\mathcal{I}}_0$, $\tilde{\xi}_{\beta,\theta}(1-c) > -1 + \exp(-b/(2\rho_D))$ for $\theta' \notin \mathcal{J}_{0,\beta}$, and $\tilde{\xi}_{\beta,\theta'}(x) \leq -3/4$ for all $\theta' \in \mathcal{J}_{0,\beta}$ and $x \in C$.

First, we show $\tilde{\xi}_{\beta,\theta}(1-c) > -1 + \exp(-b/(2\rho_D))$ for $\theta' \notin \mathcal{J}_{0,\beta}$. Obviously,

$$\xi_\beta(1/\rho_D, \theta', 1-c) \leq -1 + \exp[-b/(2\rho_D)]$$

if and only if

$$\xi_\beta(t_2(\theta'), \theta', 1-c) \leq \xi^-_\beta(1/\rho_D - t_2(\theta'), \theta' + \omega, -1 + \exp[-b/(2\rho_D)])$$

where $\xi^-_\beta(t, \theta', x) = \xi_\beta(-t, \theta', x)$. From the proof of Proposition 6.2.4 (a), we get for each $x \in C$ that $\partial_x \xi_\beta(t_2(\theta'), \theta', x) \leq \exp(-2b(1-c)(1/\rho_D - \delta_1) + 4b\delta_1)$, which is smaller than $\exp(-b/\rho_D)$, since $\delta_1 < 1/(36\rho_D)$ and $c < 1/4$. Therefore,

$$|\xi_\beta(t_2(\theta'), \theta', 1+c) - \xi_\beta(t_2(\theta'), \theta', 1-c)| \leq 2c\exp(-b/\rho_D) \leq 1/2\exp(-b/\rho_D). \tag{6.2.8}$$

In particular, this implies $|u(\theta) - \xi_\beta(t_2(\theta'), \theta', 1-c)| < 1/2 \cdot \exp(-b/\rho_D)$. As further $\xi^-_\beta(1/\rho_D - t_2(\theta'), \theta' + \omega, -1 + \exp[-b/(2\rho_D)]) \leq -1 + \exp[-b/(2\rho_D)]$ (due to Proposition 6.2.1 (b)), this yields that $\xi_\beta(t_2(\theta'), \theta', 1-c) > \xi^-_\beta(1/\rho_D - t_2(\theta'), \theta' + \omega, -1 + \exp[-b/(2\rho_D)])$ if

$$u(\theta) > -1 + \exp(-b/(2\rho_D)) + 1/2 \exp(-b/\rho_D). \tag{6.2.9}$$

Hence, $\tilde{\xi}_{\beta,\theta}(1-c) > -1 + \exp(-b/(2\rho_D))$ for $\theta' \notin \mathcal{J}_{0,\beta}$.

Given $\theta' \in \mathcal{I}_0$ with $u(\theta) = \xi_\beta(t_2(\theta'), \theta', 1) \geq -1$, there is $y \in [-1, u(\theta)]$ such that

$$\begin{aligned}\tilde{\xi}_{\beta,\theta'}(1) &= \xi_\beta(1/\rho_D - t_2(\theta'), \theta, -1) + \partial_x \xi_\beta(1/\rho_D - t_2(\theta'), \theta, y) \cdot |-1 - u(\theta)| \\ &\leq -1 + \exp(b\delta_2) \cdot |-1 - u(\theta)|\end{aligned}$$

where we used (6.1.5) and the fact that $\xi_\beta(1/\rho_D - t_2(\theta'), \theta, -1) = -1$. Thus, $\tilde{\xi}_{\beta,\theta'}(1) > -7/8$ necessarily means $u(\theta) \geq -1 + 1/8 \exp(-b\delta_2)$ which is bigger than the right-hand side of (6.2.9) for large enough b as $\delta_2 < \delta_1 \leq 1/(36\rho_D)$. Hence, $\tilde{\xi}_{\beta,\theta'}(x) \leq -3/4$ for all $\theta' \in \mathcal{J}_{0,\beta}$ and $x \in C$.

We are left to show that $\mathcal{J}_{0,\beta} \subseteq \tilde{\mathcal{I}}_{0,\beta}$, which is equivalent to showing that (6.2.9) holds for each $\theta \in T_{t_2} \setminus \tilde{T}_{t_2}$. By the above, we may assume without loss of generality that $\tilde{\xi}_{\beta,\theta'}(1) \leq -3/4$ for all $\theta' \in \tilde{\mathcal{I}}_0$ so that $(\partial^2_\Delta \xi_\beta)(t_2 - t_1, \theta' + t_1(\theta')\rho, 1) > c_0$ by the hypothesis of this claim. Note that by definition of β_+ and due to Proposition 6.2.1 (a), it follows from (6.2.8) that $u(\theta) \geq u(\theta_0 + t_2\rho) \geq -1 - 1/2 \exp(-b/\rho_D)$. Hence, for θ on the boundary of $\tilde{T}_{t_2}$, we get by means of the lower bound c_0 on the second derivatives that

$$\begin{aligned}u(\theta) &\geq u(\theta_0 + t_2\rho) + c_0 \cdot |\theta - (\theta_0 + t_2\rho)|^2 \geq -1 - 1/2 \exp(-b/\rho_D) + c_0 r_b^2 \\ &= -1 - 1/2 \exp(-b/\rho_D) + c_0 \exp(-18b\delta_1) \\ &> -1 + \exp[-b/(2\rho_D)] + 1/2 \exp(-b/\rho_D)\end{aligned}$$

for large enough b as $\delta_1 < 1/(36\rho_D)$. By means of (6.2.7), this proves (6.2.9) for all $\theta \in T_{t_2} \setminus \tilde{T}_{t_2}$. □

It remains to compute upper bounds on the first derivatives $\partial_\vartheta \xi_\beta$ and lower bounds for the second derivatives $\partial^2_\vartheta \xi_\beta$. For $\theta \in \tilde{\mathcal{I}}_0$ and $x \in C$, we have

$$\begin{aligned}\left|\partial_\vartheta \xi_\beta(t_1(\theta), \theta, x)\right| &\leq \left|(\partial_\vartheta \xi_\beta)(t_1(\theta), \theta, x)\right| + \left|\partial_t \xi_\beta(t_1(\theta), \theta, x) \cdot \partial_\vartheta t_1(\theta)\right| \\ &\leq \kappa(1+c)^2 b.\end{aligned} \tag{6.2.10}$$

This is due to the fact that $\left(\partial_\vartheta \xi_\beta\right)(t_1(\theta), \theta, x) = 0$ (see (6.1.6) and recall that $[\tilde{\mathcal{I}}_0, \tilde{T}_{t_1}] \cap B_R(\overline{\theta}) = \emptyset$) and because $\xi_\beta(t_1(\theta), \theta, x) \in C$ for all $(\theta, x) \in \mathbb{T}^d \times C$ such that

$$|\partial_t \xi_\beta(t_1(\theta), \theta, x)| = |F_\beta(\theta + t_1(\theta) + \rho, \xi_\beta[t_1(\theta), \theta, x])| \leq (1+c)^2 b. \tag{6.2.11}$$

For $t \in [t_1, t_3]$, $\theta \in \tilde{\mathcal{I}}_0$ and $x \in C$, we further have

$$\begin{aligned}
&\left|\left(\partial_\Delta \xi_\beta\right)\left(t - t_1, \theta + t_1(\theta)\rho, \xi_\beta(t_1(\theta), \theta, x)\right)\right| \\
&\leq \int_0^{t-t_1} \left|(\partial_\Delta F_\beta)\left(\theta + [s + t_1(\theta)]\rho, \xi_\beta[s + t_1(\theta), \theta, x]\right)\right| \\
&\quad \cdot \exp\left(\int_s^{t-t_1} (\partial_x F_\beta)\left(\theta + [\tau + t_1(\theta)]\rho, \xi_\beta[\tau + t_1(\theta), \theta, x]\right) d\tau\right) ds \\
&\leq \iota \cdot h''(0) \cdot b/(1 - b^{-1/2}) r_b \\
&\quad \cdot \int_0^{t-t_1} \exp\left(\int_s^{t-t_1} (\partial_x F_\beta)\left(\theta + [\tau + t_1(\theta)]\rho, \xi_\beta[\tau + t_1(\theta), \theta, x]\right) d\tau\right) ds \\
&\leq \iota \cdot h''(0) \cdot b/(1 - b^{-1/2}) \exp(5b\delta_1) r_b \leq \exp(6b\delta_1) r_b
\end{aligned} \tag{6.2.12}$$

for sufficiently large b, where we used (6.2.5) in the second step (with ι such that $|h'(y)/y| \leq \iota |h''(0)|$ for all $y \geq 0$) and (6.2.2) in the second to the last step. Observe that (6.2.12) is an upper bound on $|(\partial_\Delta \xi_\beta)(t - t_1, \theta + t_1(\theta)\rho, x)|$ for *all* $\Delta \perp \rho$ of length 1.

Now, the derivative of the map $\tilde{\mathcal{I}}_0 \times C \ni (\theta, x) \mapsto \xi_\beta\,(t - t_1 + t_1(\theta), \theta, x)$ in direction of an arbitrary $\vartheta \in \mathbb{R}^d$ with $|\vartheta| = 1$ is given by

$$\begin{aligned}
&\partial_\vartheta \xi_\beta\,(t - t_1 + t_1(\theta), \theta, x) = \partial_\vartheta \xi_\beta\left(t - t_1, \theta + t_1(\theta)\rho, \xi_\beta(t_1(\theta), \theta, x)\right) \\
&= \left(\mathrm{d}_\theta \xi_\beta\right)\left(t - t_1, \theta + t_1(\theta)\rho, \xi_\beta(t_1(\theta), \theta, x)\right) \cdot (\vartheta + \partial_\vartheta t_1(\theta)\rho) \\
&\quad + \left(\partial_x \xi_\beta\right)\left(t - t_1, \theta + t_1(\theta)\rho, \xi_\beta(t_1(\theta), \theta, x)\right) \cdot \partial_\vartheta \xi_\beta(t_1(\theta), \theta, x) \\
&= |\vartheta + \partial_\vartheta t_1(\theta)\rho| \cdot \left(\partial_\Delta \xi_\beta\right)\left(t - t_1, \theta + t_1(\theta)\rho, \xi_\beta(t_1(\theta), \theta, x)\right) \\
&\quad + \left(\partial_x \xi_\beta\right)\left(t - t_1, \theta + t_1(\theta)\rho, \xi_\beta(t_1(\theta), \theta, x)\right) \cdot \partial_\vartheta \xi_\beta(t_1(\theta), \theta, x)
\end{aligned} \tag{6.2.13}$$

where $(\mathrm{d}_\theta \xi_\beta)(t, \theta, x)$ denotes the total derivative of the map $\theta \mapsto \xi_\beta(t, \theta, x)$ (for fixed t and x) and $\Delta = (\vartheta + \partial_\vartheta t_1(\theta)\rho)/|\vartheta + \partial_\vartheta t_1(\theta)\rho|$ is indeed orthogonal to ρ. Note that due to (6.2.4), $(\partial_x \xi_\beta)(t - t_1, \theta + t_1(\theta)\rho, \xi_\beta[t_1(\theta), \theta, x]) \leq 1$ for all $t \in [t_1, 1/\rho_D]$ since $t_1(\theta) \leq t_1 + R < t_1 + \delta_1 < 1/\rho_D - 5\delta_1$ (recall that $t_1 = 1/(4\rho_D)$). By means of (6.2.10) and (6.2.12), we hence have

$$\left|\partial_\vartheta \xi_\beta\,(t - t_1 + t_1(\theta), \theta, x)\right| \leq (1 + \kappa|\rho|) \exp(6b\delta_1) r_b + \kappa(1 + c)^2 b. \tag{6.2.14}$$

These are sufficient upper bounds on the first derivatives of ξ_β with respect to Δ and ϑ. We proceed with the second derivatives.

Claim 6.2.8. $(\partial_\Delta^2 \xi_\beta)(t - t_1, \theta + t_1(\theta)\rho, \xi_\beta(t_1(\theta), \theta, x)) > \exp(b\delta_2/2)$ *for all* $\theta \in \tilde{\mathcal{I}}_0$, $t \in [t_2, t_3]$, *and* $x \in C$ *with* $\tilde{\xi}_{\beta,\theta}(x) \leq -3/4$.

Proof of the claim. As $h' \restriction_{(0,R)} < 0$ and $\partial_\Delta^2 g\left(\overline{\theta}\right) < 0$, we see by means of (6.2.6) that there is $\gamma_1 > 0$ such that for sufficiently large b we have $\partial_\Delta^2 g < -\gamma_1$ on $B_{R_0}\left(\overline{\theta}\right) \cap [\tilde{\mathcal{I}}_0, \tilde{\mathcal{I}}_0 + \omega]$, where $R_0 > 0$ is as in Claim 6.2.6. Let

$$\gamma_2 = \max_{\theta \in \mathbb{T}^D \setminus B_{R_0}\left(\overline{\theta}\right)} h''(|\theta - \overline{\theta}|)/|\theta - \overline{\theta}|^2 - h'(|\theta - \overline{\theta}|)/|\theta - \overline{\theta}|^3 \geq 0$$

and observe–again by means of (6.2.6)–that $\partial_\Delta^2 g \leq \gamma_2 r_b^2$ on $\left(B_R\left(\overline{\theta}\right) \setminus B_{R_0}\left(\overline{\theta}\right)\right) \cap [\tilde{\mathcal{I}}_0, \tilde{\mathcal{I}}_0 + \omega]$.

For θ, x, and t as in the hypothesis, we thus have

$$\begin{aligned}
&\int_0^{t-t_1} (\partial_\Delta^2 g)(\theta + [s + t_1(\theta)]\rho) \exp\left(\int_s^{t-t_1} (\partial_x F_\beta)\left(\theta + [\tau + t_1(\theta)]\rho, \xi_\beta[\tau + t_1(\theta), \theta, x]\right) d\tau\right) ds \\
&\leq \int_0^{t-t_1} \left(\gamma_2 r_b^2 \mathbf{1}_{B_R(\overline{\theta})\setminus B_{R_0}(\overline{\theta})}(\theta + [s + t_1(\theta)]\rho) - \gamma_1 \mathbf{1}_{B_{R_0}(\overline{\theta})}(\theta + [s + t_1(\theta)]\rho)\right) \\
&\quad \cdot \exp\left(\int_s^{t-t_1} (\partial_x F_\beta)\left(\theta + [\tau + t_1(\theta)]\rho, \xi_\beta[\tau + t_1(\theta), \theta, x]\right) d\tau\right) ds \\
&\leq \gamma_2 r_b^2 \exp(5b\delta_1) - \gamma_1 \exp(b\delta_2/2) \leq -\gamma_3 \exp(b\delta_2/2)
\end{aligned} \tag{6.2.15}$$

for some $\gamma_3 > 0$, where we used (6.2.2) and (6.2.3) in the second to last step (recall that $r_b = \exp(-9b\delta_1)$).

Now, plugging (6.2.12) and (6.2.2) into (6.1.9) (observe that the term with the mixed derivatives of F_β vanishes for $(*)$) yields for each $t \in [t_2, t_3]$ that

$$\begin{aligned}
&\left|\left(\partial_\Delta^2 \xi_\beta\right)\left(t - t_1, \theta + t_1(\theta)\rho, \xi_\beta(t_1(\theta), \theta, x)\right)\right| \\
&= \left|\int_0^{t-t_1} \Big[(\partial_x^2 F_\beta)(\theta + [s + t_1(\theta)]\rho, \xi_\beta[s + t_1(\theta), \theta, x])\left((\partial_\Delta \xi_\beta)[s, \theta + t_1(\theta)\rho, \xi_\beta(t_1(\theta), \theta, x)]\right)^2\right. \\
&\quad + (\partial_\Delta^2 F_\beta)\left(\theta + [s + t_1(\theta)]\rho, \xi_\beta[s + t_1(\theta), \theta, x]\right)\Big] \\
&\quad \left.\cdot \exp\left(\int_s^{t-t_1} (\partial_x F_\beta)\left(\theta + [\tau + t_1(\theta)]\rho, \xi_\beta[\tau + t_1(\theta), \theta, x]\right) d\tau\right) ds\right| \\
&\geq -2b \exp(17b\delta_1) r_b^2 - \beta b/(1 - b^{-1/2}) \\
&\quad \cdot \int_0^{t-t_1} (\partial_\Delta^2 g)(\theta + [s + t_1(\theta)]\rho) \exp\left(\int_s^{t-t_1} (\partial_x F_\beta)\left(\theta + [\tau + t_1(\theta)]\rho, \xi_\beta[\tau + t_1(\theta), \theta, x]\right) d\tau\right) ds \\
&\geq -2b \exp(17b\delta_1) r_b^2 + \gamma_3 \beta b/(1 - b^{-1/2}) \exp(b \cdot \delta_2/2)
\end{aligned}$$

which is bigger than $\exp(b\delta_2/2)$ for sufficiently large b, where we used (6.2.15) in the last step. □

Thus, the assumptions of Claim 6.2.7 are met and it remains to show that $\partial_\vartheta^2 \tilde{\xi}_{\beta,\theta}(x) > \exp(b\delta_2/4)$ for $x \in C$, $\vartheta \in \mathbb{S}^{d-1}$, and $\theta \in \mathcal{J}_{0,\beta}$. Plugging (6.2.4) into (6.1.7), yields

$$\left|\left(\partial_x^2 \xi_\beta\right)\left(t_3 - t_1, \theta + t_1(\theta)\rho, \xi_\beta[t_1(\theta), \theta, x]\right)\right| \leq 2b/\rho_D.$$

Analogously, with (6.1.8) and (6.2.12) we get

$$\left|\left(\partial_\Delta \partial_x \xi_\beta\right)\left(t_3 - t_1, \theta + t_1(\theta)\rho, \xi_\beta[t_1(\theta), \theta, x]\right)\right| \leq 2b/\rho_D \cdot \exp(6b\delta_1) r_b.$$

Finally, note that

$$\begin{aligned}\partial_\vartheta^2\xi_\beta(t_1(\theta),\theta,x) &= \left(\partial_\vartheta^2\xi_\beta\right)(t_1(\theta),\theta,x) + 2\left(\partial_t\partial_\vartheta\xi_\beta\right)(t_1(\theta),\theta,x)\cdot\partial_\vartheta t_1(\theta)\\ &\quad + \partial_t^2\xi_\beta(t_1(\theta),\theta,x)\cdot(\partial_\vartheta t_1(\theta))^2,\end{aligned} \tag{6.2.16}$$

where we used the fact that $\partial_\vartheta^2 t_1(\theta) = 0$. By means of (6.1.6), we have that $\partial_\vartheta\xi_\beta(\tau,\theta,x) = 0$ for all $\tau\in[0,1/\rho_D-\delta_1]$ so that both $\left(\partial_\vartheta^2\xi_\beta\right)(t_1(\theta),\theta,x)$ and $\left(\partial_t\partial_\vartheta\xi_\beta\right)(t_1(\theta),\theta,x)\cdot\partial_\vartheta t_1(\theta)$ vanish. Further,

$$\begin{aligned}\partial_t^2\xi_\beta(t_1(\theta),\theta,x) &= \partial_x F_\beta(\theta+t_1(\theta)\rho,\xi_\beta(t_1(\theta),\theta,x))\partial_t\xi_\beta(t_1(\theta),\theta,x)\\ &= -2b\cdot\xi_\beta(t_1(\theta),\theta,x)\cdot\partial_t\xi_\beta(t_1(\theta),\theta,x)\end{aligned} \tag{6.2.17}$$

where we used that $\mathrm{d}_\theta F_\beta(\theta+t\rho,x) = 0$ for all $t\in[0,1/\rho_D-\delta_1]$ in the first step. Since $\xi_\beta(t_1(\theta),\theta,x)\le 1+c$ and due to (6.2.11), we hence get

$$|\partial_\vartheta^2\xi_\beta(t_1(\theta),\theta,x)| \le 2\kappa^2(1+c)^3 b^2. \tag{6.2.18}$$

We are now in a position to derive a lower bound on the second derivative of $\tilde{\mathcal{I}}_0\times C\ni(\theta,x)\mapsto\xi_\beta(t_3(\theta),\theta,x) = \xi_\beta(t_3-t_1,\theta+t_1(\theta)\rho,\xi_\beta[t_1(\theta),\theta,x])$ in direction of ϑ. From (6.2.13), we get

$$\begin{aligned}&\partial_\vartheta^2\xi_\beta\,(t_3(\theta),\theta,x)\\ &= |\vartheta+\partial_\vartheta t_1(\theta)\rho|^2\cdot\left(\partial_\Delta^2\xi_\beta\right)\left(t_3-t_1,\theta+t_1(\theta)\rho,\xi_\beta(t_1(\theta),\theta,x)\right)\\ &\quad + 2|\vartheta+\partial_\vartheta t_1(\theta)\rho|\left(\partial_\Delta\partial_x\xi_\beta\right)\left(t_3-t_1,\theta+t_1(\theta)\rho,\xi_\beta(t_1(\theta),\theta,x)\right)\cdot\partial_\vartheta\xi_\beta(t_1(\theta),\theta,x)\\ &\quad + \left(\partial_x^2\xi_\beta\right)\left(t_3-t_1,\theta+t_1(\theta)\rho,\xi_\beta(t_1(\theta),\theta,x)\right)\cdot\left(\partial_\vartheta\xi_\beta(t_1(\theta),\theta,x)\right)^2\\ &\quad + \left(\partial_x\xi_\beta\right)\left(t_3-t_1,\theta+t_1(\theta)\rho,\xi_\beta(t_1(\theta),\theta,x)\right)\cdot\partial_\vartheta^2\xi_\beta(t_1(\theta),\theta,x).\end{aligned}$$

By the above computations and in particular from Claim 6.2.8, we see that for large enough b the leading term is the one containing $\left(\partial_\Delta^2\xi_\beta\right)\left(t_3-t_1,\theta+t_1(\theta)\rho,\xi_\beta(t_1(\theta),\theta,x)\right)$. This yields

$$\left|\partial_\vartheta^2\xi_\beta\,(t_3(\theta),\theta,x)\right| \ge \exp(b\delta_2/3) \tag{6.2.19}$$

for large enough b. Now, let us consider the derivatives $\partial_\vartheta^2\tilde{\xi}_{\beta,\theta}(x)$. Analogously to (6.2.13), we get

$$\begin{aligned}\partial_\vartheta\tilde{\xi}_{\beta,\theta}(x) &= \partial_\vartheta\xi_\beta\left(1/\rho_D-t_3(\theta),\theta+t_3(\theta)\rho,\xi_\beta(t_3(\theta),\theta,x)\right)\\ &= -\partial_t\xi_\beta\left(1/\rho_D-t_3(\theta),\theta+t_3(\theta)\rho,\xi_\beta(t_3(\theta),\theta,x)\right)\cdot\partial_\vartheta t_3(\theta)\\ &\quad + |\vartheta+\partial_\vartheta t_3(\theta)\rho|\cdot\left(\partial_\Delta\xi_\beta\right)\left(1/\rho_D-t_3(\theta),\theta+t_3(\theta)\rho,\xi_\beta(t_3(\theta),\theta,x)\right)\\ &\quad + \left(\partial_x\xi_\beta\right)\left(1/\rho_D-t_3(\theta),\theta+t_3(\theta)\rho,\xi_\beta(t_3(\theta),\theta,x)\right)\cdot\partial_\vartheta\xi_\beta(t_3(\theta),\theta,x)\\ &= -\partial_t\xi_\beta\left(1/\rho_D-t_3(\theta),\theta+t_3(\theta)\rho,\xi_\beta(t_3(\theta),\theta,x)\right)\cdot\partial_\vartheta t_3(\theta)\\ &\quad + \left(\partial_x\xi_\beta\right)\left(1/\rho_D-t_3(\theta),\theta+t_3(\theta)\rho,\xi_\beta(t_3(\theta),\theta,x)\right)\cdot\partial_\vartheta\xi_\beta(t_3(\theta),\theta,x),\end{aligned}$$

where we used that $F_\beta(\theta + t_3(\theta)\rho + \tau, \cdot) = 0$ for all $\tau \in [0, 1/\rho_D - t_3(\theta)]$ and $\theta \in \tilde{\mathcal{I}}_0$ in the last step. By differentiating this expression once more, we straightforwardly obtain

$$\begin{aligned}
\partial_\vartheta^2 \tilde{\xi}_{\beta,\theta}(x) &= \partial_t^2 \xi_\beta \left(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_\beta(t_3(\theta), \theta, x)\right) \cdot (\partial_\vartheta t_3(\theta))^2 \\
&\quad - 2(\partial_t \partial_x \xi_\beta)\left(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_\beta(t_3(\theta), \theta, x)\right) \cdot \partial_\vartheta t_3(\theta) \cdot \partial_\vartheta \xi_\beta(t_3(\theta), \theta, x) \\
&\quad + \left(\partial_x^2 \xi_\beta\right)\left(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_\beta(t_3(\theta), \theta, x)\right) \cdot \left(\partial_\vartheta \xi_\beta(t_3(\theta), \theta, x)\right)^2 \\
&\quad + \left(\partial_x \xi_\beta\right)\left(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_\beta(t_3(\theta), \theta, x)\right) \cdot \partial_\vartheta^2 \xi_\beta(t_3(\theta), \theta, x).
\end{aligned}$$

Let us discuss why $(\partial_x \xi_\beta)(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_\beta[t_3(\theta), \theta, x]) \cdot \partial_\vartheta^2 \xi_\beta(t_3(\theta), \theta, x)$ is the leading term. To that end, note that since $\xi_\beta(\tau, \theta + t_3(\theta)\rho, \xi_\beta[t_3(\theta), \theta, x]) < 0$ for all $\tau \in [0, 1/\rho_D - t_3(\theta)]$ and $\theta \in \mathcal{J}_{0,\beta}$, we have $(\partial_x \xi_\beta)(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_\beta(t_3(\theta), \theta, x)) \geq 1$. Together with (6.2.19), this eventually finishes the proof if we can show that the remaining terms are indeed negligible.

By an analogous computation as in (6.2.17), we see

$$\begin{aligned}
\left|\partial_t^2 \xi_\beta \left(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_\beta(t_3(\theta), \theta, x)\right)\right| &\leq 2b \cdot |\xi_\beta(1/\rho_D, \theta, x) \cdot \partial_t \xi_\beta(1/\rho_D, \theta, x)| \\
&\leq 16b^2,
\end{aligned}$$

where we used that $\xi_\beta(1/\rho_D, \theta, x) \geq -2$ (see Proposition 6.2.4 (a)) in the last step. Further,

$$\left(\partial_x \xi_\beta\right)\left(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_\beta(t_3(\theta), \theta, x)\right) \leq \exp(4b(1/\rho_D - t_3))$$

so that by putting t_3 close enough to $1/\rho_D$ (which is possible if we assume large enough b) we get small enough upper bounds on $(\partial_t \partial_x \xi_\beta)(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_\beta(t_3(\theta), \theta, x))$ (see (6.1.3)) as well as $(\partial_x^2 \xi_\beta)(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_\beta(t_3(\theta), \theta, x)) \cdot (\partial_\vartheta \xi_\beta(t_3(\theta), \theta, x))^2$ (see (6.1.7) and (6.2.14)). □

We proceed with the remaining assumptions on $\tilde{\Xi}_\beta$. Let $S > 0$ be such that

($\mathcal{A}$10) $\left|\partial_\vartheta \tilde{\xi}_{\beta,\theta}(x)\right| < S$ for all $\vartheta \in \mathbb{S}^{d-1}$ and $(\theta, x) \in \Gamma \cap \tilde{\Xi}_\beta^{-1}(\Gamma)$;

($\mathcal{A}$11) $\left|\partial_\vartheta^2 \tilde{\xi}_{\beta,\theta}(x)\right| < S^2$ for all $\vartheta \in \mathbb{S}^{d-1}$ and $(\theta, x) \in \Gamma \cap \tilde{\Xi}_\beta^{-1}(\Gamma)$;

($\mathcal{A}$12) $\left|\partial_\vartheta \partial_x \tilde{\xi}_{\beta,\theta}(x)\right| < \begin{cases} S\alpha_c & \text{for } (\theta, x) \in \mathbb{T}^d \times C \\ S\alpha_u^2 & \text{for } (\theta, x) \in \Gamma \cap \tilde{\Xi}_\beta^{-1}(\Gamma) \end{cases}$ for each $\vartheta \in \mathbb{S}^{d-1}$.

Equations (6.1.6), (6.1.9) and (6.1.8) yield that a possible choice to ensure ($\mathcal{A}$10)–($\mathcal{A}$12) for $\tilde{\xi}_{\beta,\theta}$ is to set $S = \exp(9b\delta_1)$. In case of ($\mathcal{A}$10), this can be seen from

$$\begin{aligned}
&\left|\partial_\vartheta \tilde{\xi}_{\beta,\theta}(x)\right| \\
&\overset{(6.1.6)}{\leq} \int_0^{1/\rho_D} \left|(\partial_\vartheta F_\beta)\left(s\rho + \theta, \xi_\beta(s, \theta, x)\right)\right| \exp\left(\int_s^{1/\rho_D} (\partial_x F_\beta)\left(\tau\rho + \theta, \xi_\beta(\tau, \theta, x)\right) d\tau\right) ds \\
&\leq \quad \delta_1 b/(1 - b^{-1/2}) \cdot \max_{\theta \in \mathbb{T}^D} |\partial_\vartheta g(\theta)| \exp\left(2b\delta_1\right),
\end{aligned}$$

where we used that $\partial_\vartheta F_\beta$ vanishes for $s < 1/\rho_D - \delta_1$ and that $\xi_\beta(\tau,\theta,x) \geq -1$ for each $\tau \in [0, 1/\rho_D]$ and $(\theta, x) \in \tilde{\Xi}_\beta^{-1}(\Gamma)$ due to Proposition 6.2.1 (a). However for big enough b, this expression is certainly smaller than $\exp(9b\delta_1)$. ($\mathcal{A}$11) and ($\mathcal{A}$12) can be seen in a similar fashion. Finally, we need that

$$(\mathcal{A}13)\quad \left|\partial_x^2 \tilde{\xi}_{\beta,\theta}(x)\right| < \begin{cases} \alpha_c & \text{for } (\theta,x) \in \mathbb{T}^d \times C \\ \alpha_u^2 & \text{for } (\theta,x) \in \Gamma \cap \tilde{\Xi}_\beta^{-1}(\Gamma), \end{cases}$$

which is true to, in particular, (6.1.7) and Proposition 6.2.4.

There are two more assumptions left which deal with the inverse of $\tilde{\xi}_{\beta,\theta}$.

$$(\mathcal{A}14)\quad \left|\partial_x^2 \tilde{\xi}_{\beta,\theta}^{-1}(x)\right| < \alpha_e^{-1} \text{ for each } \theta \notin \mathcal{I}_0 + \omega \text{ and } x \in E;$$

$$(\mathcal{A}15)\quad \left|\partial_\vartheta \partial_x \tilde{\xi}_{\beta,\theta}^{-1}(x)\right| < S\,\alpha_e^{-1} \text{ for each } \theta \notin \mathcal{I}_0 + \omega, x \in E \text{ and } \vartheta \in \mathbb{S}^{d-1}.$$

Observe that $\tilde{\xi}_{\beta,\theta}^{-1} = \tilde{\xi}^-{}_{\beta,\theta}$ (see equation (2.1.5) and (2.1.6)). Hence, we can derive the desired estimates for $\tilde{\xi}_{\beta,\theta}^-$ by means of (6.1.7) and (6.1.8) if we replace F_β by F_β^- and ρ by $\rho^- = -\rho$. Under the assumption of $x \in E$ and $\theta \notin \mathcal{I}_{0,\beta} + \rho$, we have that $\xi_\beta^-(t,\theta,x) \in E$ for all $t \in [0, 1/\rho_D]$ and hence $\partial_x \xi_\beta^-(t,\theta,x) \leq \exp(-2b(1-\exp[-b/(2\rho_D)]) \cdot t)$. Thus, ($\mathcal{A}$14) follows immediately for large enough b. ($\mathcal{A}$15) follows directly from the fact that $\partial_\vartheta F_\beta^-(t\rho^- + \theta, x) = 0$ and hence $\partial_\vartheta \xi_\beta^-(t,\theta,x) = 0$ for $\theta \notin \mathcal{I}_{0,\beta} + \rho$ and $t \in [0, 1/\rho_D]$.

We are now in a position to see that $(\tilde{\Xi}_\beta)_{\beta\in[0,1]}$ lies in $\mathcal{U}_\omega(\mathbb{R})$ (cf. Definition 4.2.16) if b is large enough. It is straightforward to see that $(\tilde{\Xi}_\beta)_{\beta\in[0,1]} \in \mathcal{P}_\omega(\mathbb{R})$ and that ω is Diophantine of type $(\mathscr{C}', \eta')$ (cf. Section 6.1.2). Further, with $\gamma^- = -1$ and $\gamma^+ = 1 + c$ the assumptions of Theorem 2.1.24 are easily seen to be met by $(\tilde{\Xi}_\beta)_{\beta\in[0,1]}$. It is worth mentioning that this is still true if we set $\gamma^- = -1 - \varepsilon$ for arbitrary $\varepsilon > 0$. Hence, $(\tilde{\Xi}_\beta)_{\beta\in[0,1]}$ undergoes a saddle-node bifurcation.

In order to apply Theorem 4.2.15 to $(\tilde{\Xi}_\beta)_{\beta\in[0,1]}$, we let c and δ_1 be small enough[1] so that $2b(1-c)(1/\rho_D - \delta_1) - 10b\delta_1 > b(1+c)/\rho_D$. Then, setting $\alpha = \exp(b(1+c)/\rho_D)$ and $p = 2$ ensures $\alpha_c^{-1} = \alpha_e \geq \alpha^{2/p}$ and $\alpha_l^{-1} = \alpha_u = \alpha^p$. We have just seen in this section that ($\mathcal{A}$1)–($\mathcal{A}$15) are verified by $(\tilde{\Xi}_\beta)_{\beta\in[0,1]}$. Actually, observe that ($\mathcal{A}$1)–($\mathcal{A}$15) still hold when we set the lower bound of the expanding interval E to be $-1-\varepsilon$ (for some sufficiently small $\varepsilon = \varepsilon(b) > 0$) instead of -1. Note further that we can choose α as big as we need by assuming large enough b.

Recall that in Theorem 4.2.15, we needed to assume that

$$3|\mathcal{I}_0| < \mathscr{C}'(2K_0M_0)^{-\eta'} \tag{6.2.20}$$

for some positive integers M_0 not smaller than 2 and K_0 such that $b_1 = 1 - 1/K_0$ satisfies $2b_1^2/p - 5(1-b_1^2)p = b_1^2 - 10(1-b_1^2) > 0$. Observe that–given such M_0 and K_0–(6.2.20) holds true under the assumption of small enough R (independent of b).

[1] In the case of δ_1, this essentially amounts to assuming small enough R.

Finally, we need

$$\nu = s - \mathfrak{c}\left(\alpha, b_1^2\right) S^2 \alpha^{-(b_1^2 - 10(1-b_1^2))}$$

to be positive, where $\mathfrak{c}(\alpha, b_1^2)$ is decreasing in both arguments and s is the lower bound in ($\mathscr{A}9$). Now, with S as above and $s > \exp(b\delta_2/4)$ (cf. Lemma 6.2.5), we get

$$\nu > \exp(b\delta_2/4) - \mathfrak{c}\left(\exp(b(1+c)/\rho_D), b_1^2\right) \exp\left(-b(1+c)[b_1^2 - 10(1-b_1^2)]/\rho_D + 18b\delta_1\right),$$

which is positive (for sufficiently large b) and increasing in b as long as b_1 is close to 1 and hence, as long as R is small.

Altogether, this shows: for big enough b, $(\tilde{\Xi}_\beta)_{\beta\in[0,1]}$ lies in $\mathcal{U}_\omega(X)$. Note that actually, we can guarantee $(\tilde{\Xi}_\beta)_{\beta\in[0,1]}$ to lie in $\mathcal{V}_\omega(X)$–this just amounts to assuming that b_1 is even closer to 1 and hence, that R is even smaller.

Let us fix a Diophantine rotation vector $\rho \in \mathbb{R}^D$ and only consider families of flows $\hat{\Xi}$ driven by $(t, \theta) \mapsto t \cdot \rho + \theta$ in the following. We define $\mathscr{U}_\rho(X)$ to be the set of all $\hat{F} \in \mathscr{P}(X)$ which generate families $\hat{\Xi}$ with $\hat{\tilde{\Xi}} \in \mathcal{V}_\omega(X)$. The above shows that there exists $\hat{F} \in \mathscr{U}_\rho(X)$ such that $\hat{\tilde{\Xi}} \in \operatorname{int} \mathcal{V}_\omega(X)$. Now, any C^2-small perturbation of such $\hat{\tilde{\Xi}}$ still lies in $\mathcal{V}_\omega(X)$. Since C^2-small changes of $\hat{F}$ result in C^2-small changes of $\hat{\tilde{\Xi}}$ [Wal76, §12 Satz VI], this proves that C^2-small changes of $\hat{F} \in \mathscr{U}_\rho(X)$ still lie in $\mathscr{U}_\rho(X)$. In other words, Theorem C holds true.

To close the discussion of the continuous time case, let us see how Theorem B extends to elements of $\mathscr{U}_\rho(X)$. We denote the boundary graphs of the maximal invariant set $\tilde{\Lambda}_{\beta_c}$ of $\tilde{\Xi}_{\beta_c}$ by $\phi^\pm_{\beta_c}$ and those of the maximal invariant set Λ_{β_c} of Ξ_{β_c} by $\psi^\pm_{\beta_c}$. Notice that

$$\Lambda_{\beta_c} = \Xi_{\beta_c}([0, 1/\rho_D] \times \tilde{\Lambda}_{\beta_c}) \qquad \text{and} \qquad \Psi^\pm_{\beta_c} = \Xi_{\beta_c}([0, 1/\rho_D] \times \Phi^\pm_{\beta_c}). \tag{6.2.21}$$

Let us restrict to ψ^+. As before, ψ^- can be dealt with similarly. The uniqueness of the semi-continuous representatives of $\psi^+_{\beta_c}$ and the minimality of Λ_{β_c}, and hence the fact that $\mathrm{D}_B(\Psi^+_{\beta_c}) = D + 1$, are immediate. For the Hausdorff dimension, note that $\mathrm{D}_H([0, 1/\rho_D] \times \Phi^+_{\beta_c}) = D$ because of Theorem 2.2.7 so that $\mathrm{D}_H(\Psi^+_{\beta_c}) = \mathrm{D}_H(\Xi_{\beta_c}([0, 1/\rho_D] \times \Phi^+_{\beta_c})) = D$ due to Lemma 2.2.5.

It remains to show that $\mu_{\psi^+_{\beta_c}}$ is D-rectifiable. We obviously have that $\mu_{\psi^+_{\beta_c}}$ is absolutely continuous with respect to $\mathcal{H}^D\restriction_{\Psi^+_{\beta_c}}$. Recall that by means of Proposition 5.1.2, there is an increasing sequence of sets Ω_j such that $\phi^+_{\beta_c}$ satisfies a Lipschitz condition on Ω_j and $\mathrm{Leb}_{\mathbb{T}^d}(\Omega_\infty) = 0$, where $\Omega_\infty = \mathbb{T}^d \setminus \bigcup_{j\in\mathbb{N}} \Omega_j$ (see also the proof of Theorem 5.2.1). Set $\Omega'_j = [\Omega_j, \Omega_j + \omega]$ and $\Psi^+_{\beta_c}\restriction_{\Omega'_j} = \Psi^+_{\beta_c} \bigcap \Omega'_j \times X$ for $j \in \mathbb{N} \cup \{\infty\}$. Note that $\mathrm{Leb}_{\mathbb{T}^D}(\Omega'_\infty) = 0$. Hence, $\mu_{\psi^+_{\beta_c}}(\Psi^+_{\beta_c}\restriction_{\Omega'_\infty}) = 0$ so that $\mu_{\psi^+_{\beta_c}}$ is also absolutely continuous with respect to $\mathcal{H}^D\restriction_{\Psi^+_{\beta_c}\setminus\Psi^+_{\beta_c}\restriction_{\Omega'_\infty}}$. Since $\Psi^+_{\beta_c} \setminus \Psi^+_{\beta_c}\restriction_{\Omega'_\infty} \subseteq \bigcup_{j\in\mathbb{N}} \Psi^+_{\beta_c}\restriction_{\Omega'_j}$ and as $\Psi^+_{\beta_c}\restriction_{\Omega'_j}$ is the image of the Lipschitz continuous function $[0, 1/\rho_D] \times \Omega_j \ni (\tau, \theta) \mapsto \Xi_{\beta_c}(\tau, \theta, \phi^+_{\beta_c}(\theta))$, we get the rectifiability of $\mu_{\psi^+_{\beta_c}}$ (cf. the proof of Theorem 5.2.1).

A. Dynamical systems

We deal with certain classes of dynamical systems in this thesis. Accordingly, we assume some background in dynamical systems theory throughout this work. Nevertheless, for the sake of a self-contained exposition, we provide a few basic definitions in this appendix which the reader may find useful to consult.

Basically, the results of this thesis are located between two branches of dynamical systems theory: topological dynamics (see the next paragraph) and ergodic theory (dealt with in the last paragraph). There is a vast literature on both topics, but to name at least one standard reference for each branch, the interested reader is referred to [Aus88] and [Wal82], respectively. Further, a standard reference for a broad variety of topics in dynamical systems in general is [KH97].

A.1. Topological dynamical systems

We do not aim for the most general notion of dynamical systems here. In the following, Θ is always assumed to be a metrizable space.

Definition A.1.1. A *(discrete time topological) dynamical system (on* Θ*)* is a continuous map $f\colon \Theta \to \Theta$. Given $\theta \in \Theta$, we call $\mathcal{O}(\theta) = \{f^n(\theta)\colon n \in \mathbb{N}\}$ the *orbit of* θ *under* f.

Remark. Slightly abusing notation, we may denote *full orbits* $\{f^n(\theta)\colon n \in \mathbb{Z}\}$ by $\mathcal{O}(\theta)$, too, if $f^n(\theta)$ is well-defined for all $n \in \mathbb{Z}$.

Definition A.1.2. A *(continuous time topological) dynamical system (on* Θ*)* or *flow* is a continuous mapping $\rho\colon \mathbb{R} \times \Theta \to \Theta$ satisfying

$$\rho(0,\theta) = \theta \qquad \text{and} \qquad \rho(t,\rho(s,\theta)) = \rho(t+s,\theta),$$

for all $t, s \in \mathbb{R}$. It is customary to denote a flow as a family $(\rho_t)_{t\in\mathbb{R}} \in \mathbb{R}$ (or simply ρ_t) of self-mappings on Θ. Given $\theta \in \Theta$, the set $\mathcal{O}(\theta) = \{\rho_t(\theta)\colon t \in \mathbb{R}\}$ is called the *orbit of* θ under the flow ρ.

Definition A.1.3. Suppose Θ is a compact metrizable space. Then a topological (either continuous time or discrete time) dynamical system is called *minimal* if for every $\theta \in \Theta$ the orbit $\mathcal{O}(\theta)$ is dense in Θ, that is, $\overline{\mathcal{O}(\theta)} = \Theta$.

If Θ is a smooth manifold, a natural way to define a flow is by means of initial value problems. In general, the thus generated "flow" might not be defined for all times t and points θ (cf. [KH97, Section 0.2]).

To be more precise, let us consider the initial value problems

$$\dot{\rho}(t, x) = F(\rho(t, x)), \qquad \rho(0, x) = x \ (x \in X), \tag{A.1.1}$$

where $X \subseteq \mathbb{R}$ is some non-degenerate open interval. If F is continuous, it is well-known that for each x there exists a unique solution $\rho(\cdot, x)$ of (A.1.1) depending continuously on x, where $\rho(t, x)$ is defined for all t in some open interval which we may assume to be maximal (cf. [Har64, Chapter V, Theorem 2.1]). As this maximal interval not necessarily equals the whole real line, the following definition is useful.

Definition A.1.4. A *local (continuous time topological) dynamical system (on a metrizable space* Θ*)* or *local flow* is a continuous mapping $\rho\colon U \subseteq \mathbb{R} \times \Theta \to \Theta$ with the following properties.

- The domain U is such that for each θ in Θ the set $U_\theta = \{t \in \mathbb{R} \colon (t, \theta) \in U\}$ is a non-degenerate interval containing 0 with $U_{\rho(s,\theta)} = U_\theta - s$ for all $s \in U_\theta$.
- For each $\theta \in \Theta$, $s \in U_\theta$, and $t \in U_{\rho(s,\theta)}$ we have

$$\rho(0, \theta) = \theta \qquad \text{and} \qquad \rho(t, \rho(s, \theta)) = \rho(t + s, \theta).$$

Slightly abusing notation, we may–as before–denote a local flow by ρ_t, too.

Definition A.1.5. Given a dynamical system (possibly local in the sense of the previous definition) on Θ, a subset $M \subseteq \Theta$ is called *minimal* if the restriction of the system to M, that is, $f\restriction_M$ or $\rho_t\restriction_M$, is minimal in the sense of Definition A.1.3.

Remark. Note that even if we start with an a priori local flow, its restriction to a given minimal set has to be a complete flow in the sense of Definition A.1.2 in order to fit into the above definition.

The following statement is an implication of Zorn's Lemma (cf. [KH97, Proposition 3.3.6][1]).

Proposition A.1.6. *Every topological dynamical system in the sense of Definition A.1.1 and Definition A.1.2, respectively, on a compact metrizable space* Θ *has a minimal set.*

A.2. Ergodic theory

Ergodic theory studies statistical properties of measure preserving dynamical systems. For brevity, we restrict to discrete time systems in this section as the adaption to flows is standard. Also note that all measures under investigation are supposed to be probability measures.

[1] Observe that the proof provided in [KH97] carries over literally to the continuous time case.

Definition A.2.1. A topological dynamical system f is said to *preserve* a measure m on Θ, which is hence called *invariant*, if $m \circ f^{-1} = m$.

Theorem A.2.2 ([KH97, Theorem 4.1.1])**.** *A continuous map on a metrizable compact space has an invariant measure.*

Definition A.2.3. A measure preserving system is said to be *ergodic* (with respect to an invariant measure m) if for every set A with $f^{-1}(A) = A$, $m(A)$ is either 1 or 0. In this case, we also say *m is ergodic*. A topological dynamical system on a compact, metrizable space is called *uniquely ergodic* if it allows for a unique invariant measure.

Proposition A.2.4 ([KH97, Proposition 4.1.8])**.** *A uniquely ergodic system is ergodic.*

One of the most important theorems in Ergodic Theory is the following [KH97, Theorem 4.1.2 & Corollary 4.1.9].

Theorem A.2.5 (Birkhoff's Ergodic Theorem)**.** *Suppose f is ergodic with respect to a measure m and $h\colon \Theta \to \mathbb{R}$ is integrable. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} h(f^{\ell}(\theta)) = \int_{\Theta} h\, dm$$

m-almost surely (a.s.).

Under additional assumptions on the function h (and the f-invariant measures) we even get uniform convergence in the above equation.

Theorem A.2.6 (cf. [Her83, Lemma in Section 5.2] & [SS00, Theorem 1.2])**.** *Suppose $f\colon \Theta \to \Theta$ is a continuous map on a compact metrizable space Θ and h is a continuous function with $a \in \mathbb{R}$ such that*

$$\int h\, dm = a$$

for all f-invariant measures m. Then

$$\frac{1}{n} \sum_{\ell=0}^{n-1} h\left(f^{\ell}(\theta)\right) \xrightarrow{n\to\infty} a$$

uniformly in Θ.

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Jena, September 2015 Gabriel Fuhrmann